OR Spektrum (1997) 19: 123–129

OR Spektrum

Pure and mixed strategies for the EOQ repair and waste disposal problem

Knut Richter

Europa-Universität Viadrina, Große Scharrnstr. 59, D-15230 Frankfurt, Germany
(Tel. 0335-5534 423, Fax 0035-5534 675, e-mail: richter@euv-frankfurt-o.de)

Received: 7 March 1996 / Accepted: 20 September 1996

Abstract. In this paper the analysis of the EOQ repair and waste disposal model with variable setup numbers for production and repair within some collection time interval is continued. The cost analysis is now extended to the extreme waste disposal rates and it is shown that the pure (bang-bang) policy of either no waste disposal (total repair) or no repair (total waste disposal) dominates the strategy of mixing waste disposal and repair. Moreover, the different behavior of the minimum cost, of the optimal setup numbers, lot sizes and collection intervals for small and large waste disposal rates is discussed.

Zusammenfassung. In diesem Beitrag wird die Analyse des EOQ-Reparatur- und Entsorgungsproblems mit variablen Loszahlen für die Produktion und die Reparatur innerhalb eines Sammelzeitintervalls fortgesetzt. Die Kostenanalyse wird hier auf extreme Werte der Entsorgungsrate ausgedehnt und es wird gezeigt, daß die reine (bang-bang) Politik des vollständigen Verzichts auf Entsorgung (vollständige Reparatur) oder auf Reparatur (vollständige Entsorgung) die Mischstrategie von Entsorgung und Reparatur dominiert. Weiterhin wird das unterschiedliche Verhalten der minimalen Kosten, der optimalen Loszahlen, Losgrößen und Sammelzeitintervalle für geringe und große Entsorgungsraten diskutiert.

Key words: Production, EOQ model, waste disposal, cost minimization, remanufacturing

Schlüsselwörter: Produktion, Losgrößenmodell, Entsorgung, Kostenminimierung, Wiederverwendung

1. Introduction

1.1. The EOQ waste disposal model

The problem of tracing the interaction between economic and ecological factors to production is here discussed for the following simplified situation: Let a producer have the facilties to produce *new goods and to remanufacture used goods*. He therefore might take back his used output and remanufacture it, or to *dispose of* the returned used products. The fraction $\alpha \in [0, 1]$ of the disposed of (not remanufactured) used products per output has been called in [10–13] *waste disposal rate* and this notion will be used here, having in mind that the rate $\alpha \cdot d$ from the per period demand d is disposed of. This rate can be also regarded as a (*mixed*) *strategy* combining the pure strategies of *total repair* (and no waste disposal) and *total waste* disposal (no repair) and at the same time as a measure of *ecological* (green) *behavior*: a low rate will contribute to the development of cyclic production structures. On the opposite, high waste disposal rates imply increasingly large environmental cost to the society, and perhaps more and more to the producers themselves.

It seems to be important to know if, and under which conditions, the (per unit) price e for the waste disposal to be paid by the producer is relevant to the production decisions. This problem of tracing the reaction of optimal decisions to changing ecologically relevant inputs has been studied in [14] for general linear programming models. Here, more concretely, the producer is assumed to choose some production decision variable X as for instance, the lot size. The related cost $G(X, \alpha, e)$ is considered as a function of the action X, of the waste disposal rate a and of the waste disposal price e. The producer is supposed to minimize the function $G(X, \alpha, e)$, i. e. he is interested in solving the problem $G(X, \alpha, e) \to \min_{X,\alpha}$.

Two problems might be then worth studying:

(i) [ECOL $\to$ ECON]: Tracing the economical consequences of ecological behavior.

For various fixed e the functions $g(\alpha, e) = \min_X G(X, \alpha, e)$ and the *optimal solutions* $X(\alpha, e) = \arg\min_X G(X, \alpha, e)$ are to be determined. These functions show how the minimal cost and the optimal production variable X react, when the ecological attitude α, or, in other words, the mixed strategy, changes.

(ii) [ECON $\to$ ECOL]: Tracing the ecological consequences of economic pressure.

The function $\alpha(e) = \arg\min_\alpha g(\alpha, e)$ explains how a cost minimizer determines his ecological attitude α with respect to changing waste disposal prices. This economically *optimal* strategy is either pure or mixed, i. e. $\alpha(e) \in \{0\} \cup \{1\} \cup (0, 1)$. From the "green" point of view perhaps the optimality of $\alpha(e) \ll 1$ is preferred.

Both problems will be regarded here for the case of the deterministic two stage economic order quantity-problem [10–13] modeling the production of new and the repair of used products as for instance, containers (comp. [7]), in a first shop and the employment of the products in a second shop. The used products can either be stored at the second shop and then be brought back at the end of the collection interval $[0, T]$ to the first shop for repair, or be disposed of somewhere outside. In the first shop lot sizes of newly manufactured products and of repairable products have to be determined in order to meet a constant demand rate of the second shop. Some of the used products are collected at the second shop according to a not necessarily unique *repair rate*. The share of the products not provided for repair is again called waste disposal rate. In the model not only the demand for new or repaired products is assumed to be deterministic, but also the return flow of used products, the stochastic nature of which has been studied extensively in [8, 9], is deterministic and can be controlled by the producer. On one hand, this restriction narrows, of course, the practicability of the approach. On the other hand, the explicit results and properties provided in the paper, perhaps, can be only obtained for restricted problems.

The *lot size* x, the *setup number* m of repair lots and the *setup number* n of production lots within the variable *collection interval* make the decision variable X of the producer. His per time unit cost function $G(x, m, n, \alpha)$ consists of a sum of two different functions $G(x, m, n, \alpha) = K(x, m, n, \alpha) + R(\alpha, e)$ which is to be minimized. The first function $K(x, m, n, \alpha)$ covers the cost factors which are related to the traditional *EOQ-framework*. The other function $R(\alpha, e)$ arises outside the EOQ-framework and includes the additional *linear repair, production* and especially the *waste disposal cost* associated with e, in other words, the *non-EOQ-related cost*.

The problem [ECOL $\to$ ECON] reduces now for the EOQ repair and waste disposal problem to the determination of the optimal x, m, n for given α. Since the function $R(\alpha, e)$ does not depend on these variables, the function $K(x, m, n, \alpha)$ is to be minimized. In [12] this problem has been solved by a two-stage method, first by finding the optimal lot size $x(m, n, \alpha)$ and then, after replacing x in $K(x, m, n, \alpha)$, by determining the optimal set up numbers $m(\alpha)$ and $n(\alpha)$. Then, using the notation $x(\alpha) = x(m(\alpha), n(\alpha), \alpha)$ the decision variables $X(\alpha, e) = \{x(\alpha), m(\alpha), n(\alpha)\}$ show the reaction to changes of α. Similarly, the function $g(\alpha, e)$ can be transformed into $g(\alpha, e) = K(\alpha) + R(\alpha, e)$ with $K(\alpha) = K(x(\alpha), m(\alpha), n(\alpha))$.

This function $g(\alpha, e)$ has been proved to be *convex* for smaller waste disposal rates and *concave* for the variable a in [12] if the setup numbers are not necessarily integers. Here this analysis is continued by exploring the reasons for these properties. Furthermore it will be shown, that – if it is technologically feasible – the problem [ECON $\to$ ECOL] has an extreme solution: the cost minimal waste disposal rate is always $\alpha^* = 0 \vee 1$, i. e. either no waste disposal (total repair) or no repair (total waste disposal) is optimal. At the end the different behavior of the optimal setup numbers, lot sizes and collection intervals for small and large waste disposal rates is analyzed.

The problem of the interaction of economic and ecological factors has found some attention in last years. General ideas of modeling these problems are provided in [2–4, 14]. More practical oriented are the mainly stochastic models studied in [1, 6–9]. The problems of the environmental management are discussed informally in [5, 15] and many other papers. The EOQ repair and waste disposal model has been introduced by the author in [10] and extended in [11–13].

In the next Sect. 1.2. the model is described formally. In Sect. 2, firstly the analysis of an auxiliary problem introduced in [12] is continued, secondly the structure of the optimal setup numbers is given and thirdly some explanation is provided for the existence of regions of different (convex-concave) behavior of the minimum cost. It will be seen, that on one hand, this behavior depends on the relationship between the repair rate, the waste disposal rate, the total repair minimum cost ($\alpha = 0$) and total waste disposal minimum cost ($\alpha = 1$). On the other hand, these regions are related to the (cost oriented) preference of repair to disposal, or disposal to repair. This preference relation is changing, if the model inputs change. In the third section the optimality of the pure strategies is proved. In the fourth section the properties of the other parameters of the problem are described. The paper ends with some conclusions.

1.2. The formal description of the EOQ repair and waste model

The model is based on the following assumptions: A first shop is providing a homogeneous product used by a second shop at a constant demand rate of d items per time unit. The first shop is manufacturing new products and it is also repairing products used by the second shop.

The repaired products are then regarded as new. The products are employed by the second shop and collected there according to a repair rate β. The other products are immediately disposed as waste outside according to the waste disposal rate $\alpha = 1 - \beta$. At the end of some variable time interval $[0, T]$ the collected products are brought back to the first shop and will be stored as long as necessary and then repaired. If the repaired products are finished the manufacturing process starts to cover the remaining demand for the time interval.

The processes of manufacturing, repairing and using the products are supposed to be instantaneous. The inventory stocks occurring in this system are illustrated by Fig. 1. Note that in the case of $m > 1$ additional stocks of used products not displayed in the figure occur in the first shop.

The following cost inputs will be used: The repair setup cost $r > 0$, the production setup cost $s > 0$, the per unit cost/prices $b, k > 0$, and $e \in (-\infty, +\infty)$ for manufacturing,

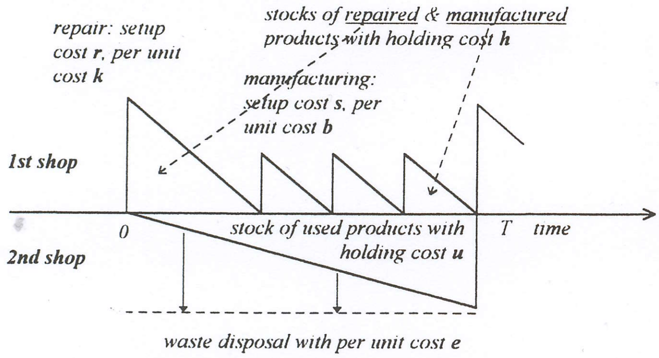

Fig. 1. Cost inputs and inventory stocks for the setup numbers $m = 1$ and $n = 3$

repairing and disposing products and the per unit per time unit holding cost h, $u > 0$ at first and second shop, respectively. The notations of the waste disposal rate and repair rate α, β, $\alpha + \beta = 1$, $0 < \alpha, \beta < 1$, of the demand rate $d > 0$, of the length T of the collection interval, of the lot size $x = dT$ of the collection interval and of the *setup numbers* $n, m \in \{1, 2, \dots\}$ for production and repair are applied to formulate the models. If these variables are fixed then the demand of the second shop is satisfied by repairing $\beta x = \beta dT$ units in m lots of size $\beta x/m$ and by producing αx units of new items in n lots of size $\alpha x/n$.

First, only the EOQ-related setup cost and the holding cost parameters are considered. Then the over all cost for a collection interval $[0, T]$ is

$$K_z = (mr+ns) + h\left(\frac{\alpha^2 x^2}{n} + \frac{\beta^2 x^2}{m}\right)/2d + u\beta T x/2 + u\beta^2 x^2 (m-1)/2dm,$$

where the fixed cost $(mr + ns)$, the holding cost $h(\alpha^2 x^2/n + \beta^2 x^2/m)/2d$ for new and repaired products and the holding cost $u\beta Tx/2 + u\beta^2 x^2 (m-1)/2dm$ for used products in the second and first shop, respectively, are included. The per time unit cost is then

$$K(x, m, n, \alpha) = K_z/T = d(mr+ns)/x + \frac{x}{2}[(\alpha^2/n + \beta^2/m)h + u\beta + u\beta^2(m-1)/m], \quad (1)$$

with $H(m, n, \alpha) = (\alpha^2/n + \beta^2/m)\, h + u\beta + u\beta^2 (m-1)/m$ as total per time unit per unit holding cost.

Let now the non-EOQ-related cost inputs be included. The sum of linear manufacturing cost, waste disposal cost and repair cost per time unit is given by the function $R(\alpha, e)$ with

$$R(\alpha, e) = d(\alpha(b+e) + (1-\alpha)k) = d(\alpha(b+e-k) + k). \quad (2)$$

Hence the overall per time unit cost is

$$G(x, m, n, \alpha) = K(x, m, n, \alpha) + R(\alpha, e), \quad (3)$$

and the corresponding optimal lot sizes, setup numbers and waste disposal rate have to be determined.

Before discussing the general problem the pure situations of no waste disposal (total repair) and no repair (total disposal) will be considered shortly:
(i) If no product will be disposed of, all of them must be collected and repaired. Then every product will be stored twice, as a repaired product and as a used product. The per time unit cost is then given by $K(x, 1, 0, 0) = dr/x + (h + u)x/2$, and the total repair minimum cost is $K_0 = \sqrt{2dr(h+u)}$. With the additional repair cost a cost expression

$$g(0, e) = K_0 + dk \quad (4)$$

occurs. (ii) If no product is repaired the situation appears which is usually treated in the literature. The per time unit cost is $K(x, 0, 1, 1) = ds/x + hx/2$ and the total disposal minimum cost is $K_1 = \sqrt{2dsh}$. With the additional manufacturing and waste disposal cost the expression

$$g(1, e) = K_1 + d(b+e) \text{ is found.} \quad (5)$$

2. The optimal solution for the EOQ repair and waste disposal model

2.1. *Introducing the auxiliary problem*

The auxiliary problem, studied below, has been introduced in [12]. It helps to understand the structure of the solution of the original problem. Therefore it is discussed here once more. If the cost function $K(x, m, n, \alpha)$, which is obviously convex and differentiable in x, is to be minimized in $x > 0$ for fixed $m, n \geq 1$ and α, then the cost minimal lot sizes $x(m, n, \alpha)$ can be derived from $\delta K/dx = 0$. Then $x(m, n, \alpha) = \sqrt{\frac{2d(mr+ns)}{H(m,n,\alpha)}}$ and the minimal cost is

$$K(m, n, \alpha) = \sqrt{2d(mr+ns)\, H(m, n, \alpha)}. \quad (6)$$

Now the optimal $m(\alpha)$ and $n(\alpha)$ for $m, n \geq 1$ and the value $K(\alpha) = K(m(\alpha), n(\alpha), \alpha)$ have to be found. Instead of the function (6) however the function

$$S(m, n, \alpha) = (mr+ns) \cdot ((\alpha^2/n + \beta^2/m)h + \beta u + \beta^2 u(m-1)/m), \quad (7)$$

can be analysed. For both functions the relationship

$$K(m, n, \alpha) = \sqrt{2d \cdot S(m, n, \alpha)} \quad (8)$$

holds. The parameters in the function (7) can be replaced by

$$A = rh\alpha^2, \; B = s(h-u)\beta^2, \; C = ru(\beta + \beta^2), \; D = su(\beta + \beta^2), \; E = sh\alpha^2 + r(h-u)\beta^2 \quad (9)$$

and the function

$$S(m, n) = A\frac{m}{n} + B\frac{n}{m} + Cm + Dn + E, \quad (10)$$

appears, where the parameters (9) fulfill

$$A, C, D, B + D, E > 0. \quad (11)$$

Let the parameters A, B, C, D, E be arbitrary real numbers, i. e. the restriction (11) associated with the inputs of the original problem, is left. The auxiliary problem of minimizing this continuous (nonconvex) function (10)

$$S(m, n) \rightarrow \min, \ (m, n) \in R = \{(m, n): (m, n) \geq 1\}, \qquad (12)$$

i. e. the problem of finding an optimal (m, n) is studied below. The problem is called solvable, if there exists an optimal solution. For negative inputs C or D, for instance, the problem is not solvable.

2.2. *The properties of the auxiliary and the original problems*

The problem (12) has been solved in [12] for the assumption (11). It is easy to see that the Theorem provided below holds for all real inputs. The formulae appear due to the properties (comp. [10]) that the optimal solutions are on the line $n = 1$ if $B \geq A$ and on $m = 1$ if $B \leq A$ and that the function $S(m, n)$ is convex on these lines. In other words, the objective function is monotonously increasing in the first case in n, and in the second case in m.

Theorem 1. *If the problem* (12) *is solvable there are three cases of optimal solutions* (m, n) *and minimum cost expressions* $S^* = \{S_i^*\}_{i=1}^3$ *for the function* (10):

(i) $B \geq A + C \Rightarrow (m^*, n^*)$

$$= \left(\sqrt{\frac{B}{A+C}}, 1\right), \quad S_1^* = 2\sqrt{B(A+C)} + D + E,$$

(ii) $A - D \leq B \leq A + C \Rightarrow (m^*, n^*)$

$$= (1, 1), \quad S_2^* = A + B + C + D + E \qquad (13)$$

(iii) $A \geq B + D \Rightarrow (m^*, n^*)$

$$= \left(1, \sqrt{\frac{A}{B+D}}\right), \quad S_3^* = 2\sqrt{A(B+D)} + C + E.$$

Remarks. 1) The structure of the optimal solution and of the minimum value for the problem (12) is symmetrical with respect to the cases (i) and (iii), i. e. if A and B, and C and D, respectively, are replaced one by the other, then case (i) turns to case (iii), and vice versa.
2) The optimal solutions do not depend on the inputs D and E in case (i) and on the inputs C and E in case (iii). These inputs occur as constants in the expressions for the minimum value.
3) The conditions of (i) and (iii) turn the objective function (10) up in the direction of rising m and n, respectively.

The application of Theorem 1 provides the following properties of the optimal solution and the minimal values of the original problem:

Theorem 2 (Richter [12]): *For the function* (7) *the optimal solution* $(m(\alpha), n(\alpha))$, *the lot size* $x(\alpha)$ *and the minimal value* $K(\alpha) = \{K_i(\alpha)\}_{i=1}^3$ for $0 < \alpha < 1$ *are given by*

(i) $\{h > u\} \wedge \{\beta^2 s(h-u) \geq r(\alpha^2 h + \beta u(1+\beta))\}$

$$\Rightarrow m(\alpha) = \beta\sqrt{\frac{s(h-u)}{r(\alpha^2 h + \beta u(1+\beta))}}, \quad n(\alpha) = 1,$$

$$x(\alpha) = \sqrt{\frac{2ds}{\alpha^2 h + \beta u(1+\beta)}}$$

and

$$K_1(\alpha) = \sqrt{2d}\left(\beta\sqrt{r(h-u)} + \sqrt{s(\alpha^2 h + \beta u(1+\beta))}\right)$$

(ii) $rh\alpha^2 - su\beta \leq sh\beta^2 \leq r\left(\alpha^2 h + \beta u(1+\beta)\right) + us\beta^2$

$$\Rightarrow (m(\alpha), n(\alpha)) = (1, 1),$$

$$x(\alpha) = \sqrt{\frac{2d(r+s)}{(\alpha^2+\beta^2)h + u\beta}} \qquad (14)$$

and $K_2(\alpha) = \sqrt{2d(r+s)(h(\alpha^2+\beta^2) + \beta u)}$

(iii) $\alpha^2 rh \geq s\beta(\beta h + u) \Rightarrow m(\alpha) = 1,$

$$n(\alpha) = \alpha\sqrt{\frac{rh}{s\beta(\beta h + u)}}, \quad x(\alpha) = \sqrt{\frac{2dr}{\beta(\beta h + u)}},$$

$$K_3(\alpha) = \sqrt{2d}\left(\alpha\sqrt{sh} + \sqrt{r\beta(\beta h + u)}\right).$$

Proof: Let only the case (iii) be shortly explained. Due to the case (iii) from Theorem 1 and formula (8)

$$\frac{(K_3(\alpha))^2}{2d} = 2\sqrt{A(B+D)} + C + E$$

$$= 2\sqrt{rh\alpha^2(s(h-u)\beta^2 + su\beta(1+\beta))} + ru\beta(1+\beta)$$

$$+ sh\alpha^2 + r(h-u)\beta^2 = 2\sqrt{sh\alpha^2 r\beta(h\beta+u)} + sh\alpha^2$$

$$+ r\beta(h\beta+u) = \left(\sqrt{sh\alpha^2} + \sqrt{r\beta(h\beta+u)}\right)^2 \text{ holds.} \quad \square$$

Remark. The proof makes clear that the parameters A,…, E are related in such a way that the minimum cost separates into two independent values in the cases (i) and (iii), respectively, which can be treated as minimum cost for some special EOQ-models.

According to [12] the function $K(\alpha)$ is continuously differentiable and the conditions (i) – (iii) divide the interval (0, 1) of the waste disposal rates into three regions $(0, \alpha_1]$, $[(\alpha_1, \alpha_2]$, $[\alpha_2, 1)$ with different behavior of the cost functions $K_i(\alpha)$. While under the realistic assumption of $4h(h+u) \geq u^2$ the function is convex in the second region, it is always convex in the first and concave in the third one. In terms of the problem (1) – (3) the function $g(\alpha, e)$ is then provided by $g_i(\alpha, e) = K_i(\alpha) + d(\alpha(b+e) + (1-\alpha)k)$, $i = 1, 2, 3$, and it is obviously also convex-concave. As an example, such a function and the reaction of the setup number $n(\alpha)$ to α are illustrated in Fig. 2. The

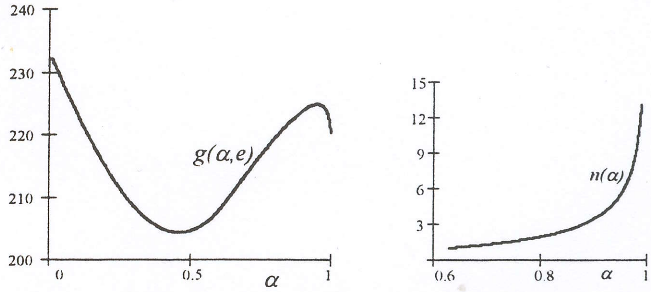

Fig. 2. The functions $g(\alpha, e)$ and $n(\alpha)$ for $s = 140$, $r = 100$, $h = 10$, $u = 4$, $d = 5$, $k = 10$, $e + b = 20$ with $\alpha_1 = 0.077$ and $\alpha_2 = 0.631$

values for $m(\alpha)$ are here near one, therefore they are not presented graphically.

Remark. Since the waste disposal rate α appears only in A, while due to the holding of used products the repair rate β is present in the other parameters, the inputs A, B, C, D in (9) are nonsymmetrical with respect to α and the structure of the optimal solution and of the minimum cost is obviously not symmetrical as in the case of the auxiliary problem (12).

It will be seen in Lemma 4 of Sect. 4 that the situation of the cases (i) $m(\alpha) > 1, n(\alpha) = 1$ and (iii) $m(\alpha) = 1, n(\alpha) > 1$ can be treated as a (cost oriented) *preference* of repair to disposal and of disposal to repair, respectively. In other words, such preferences occur if $\alpha < \alpha_1$ or $\alpha_2 < \alpha$. If $n(\alpha) = 1$ and $m(\alpha) \geq 1$, or $m(\alpha) = 1$ and $n(\alpha) \geq 1$, respectively, i.e. if $\alpha \leq \alpha_2$ or $\alpha_1 \geq \alpha$, then it makes sense to speak of weak preference of repair to disposal (or disposal to repair).

2.3. *Interpretation of the properties of the minimum cost function*

2.3.1. The appearance of regions of different behavior. Now the results of Theorem 2 will be discussed in greater detail:

a) If $u = 0$ then the three cases of Theorem 2 reduce to

$$\text{(i)}\quad \beta^2 s \geq \alpha^2 r \Rightarrow m(\alpha) = \frac{\beta}{\alpha}\sqrt{\frac{s}{r}},\quad n(\alpha) = 1,$$

$$\text{(ii)}\quad \beta^2 s \leq \alpha^2 r \Rightarrow m(\alpha) = 1, n(\alpha) = \frac{\alpha}{\beta}\sqrt{\frac{r}{s}}.$$

The minimum cost is $K(\alpha) = \sqrt{2dh}\left(\alpha\sqrt{s} + \beta\sqrt{r}\right)$ in both the cases. The three sets of waste disposal rates appear with $\alpha_1 = \alpha_2 = \frac{\sqrt{s}}{\sqrt{s} + \sqrt{r}}$. The setup number n equals one, or, in other words, repair is preferred weakly to disposal, if and only if

$$\frac{\beta}{\alpha} \geq \sqrt{\frac{r}{s}}\left(= \frac{K_0}{K_1}\right) \tag{15}$$

holds. The left-hand side expresses the ratio of repaired products to disposed products, called here the *repaired product quantity ratio*. The right-hand side gives the ratio of the total repair minimum cost K_0 to the total disposal minimum cost K_1, called the *repair minimum cost ratio*. Hence, repair is weakly preferred to disposal, if and only if the repaired product quantity ratio is not less than the repair minimum cost ratio. If to look at objective function from a geometrical point of view, condition (15) turns the function up in the direction of rising n.

The minimum cost appears as the linear combination of the total repair minimum cost and of the total disposal minimum cost and therefore is a linear function of α. If every value $\alpha \in [0, 1]$ is feasible, the optimal waste disposal rate is determined by that α^* which minimizes

$$g(\alpha, e) = K(\alpha) + R(\alpha, e)$$
$$= \sqrt{2dh}\left(\alpha\sqrt{s} + \beta\sqrt{r}\right) + d(\alpha(b + e - k) + k).$$

It is then clear that $\alpha^* = 0$ if

$$\sqrt{2dhr} + dk < \sqrt{2dhs} + d(b + e) \quad\text{and}\quad \alpha^* = 1$$

in the opposite case.

b) Let the more complicated case $h \geq u > 0$ be considered. It is easy to see that the inequality $B \geq A$ is, in terms of (9), equal to

$$\beta^2 s\,(h-u) \geq r\,\alpha^2. \tag{16}$$

This inequality is always fulfilled for sufficiently small values of α (and large β) and it implies that $n = 1$. This inequality covers the two sets $[0, \alpha_1]$ and $[\alpha_1, \alpha_2]$. The following Lemma, which can be proved elementary, transforms the inequality (16) into such α relation which can be treated in economic terms.

Lemma 1. *Let $h > u > 0$ be fulfilled. Then the inequality (16) holds if and only if*

$$\frac{\beta}{\alpha} \geq \frac{\sqrt{rh}}{\sqrt{s(h-u)}}\left(= \rho \cdot \frac{K_0}{K_1}\right) \text{ with } \rho = \frac{h}{\sqrt{h^2 - u^2}}.$$

The inequality (17) can be treated in the following way: While the repaired product quantity ratio is on the left hand side, the right-hand side covers the repair minimum cost ratio, corrected by the parameter $\rho > 1$. This parameter expresses the relationship between the two holding cost inputs and it is a monotonically increasing function of u. Hence, repair is weakly preferred to disposal if and only if the repaired product quantity ratio is not less than the corrected repair minimum cost ratio.

Using similar arguments we can show that disposal is weakly preferred to repair if and only if the repaired product quantity ratio is not greater than the (corrected) repair minimum cost ratio.

c) The case $u \geq h$ is not so much of interest, since only the solutions of type (ii) and (iii) occur in Theorem 2, and, disposal is always weakly preferred to repair.

2.3.2. The convex-concave behavior of the minimum cost. Now, the problem of the different behavior of the minimum cost function at the different regions will be discussed. Due to the remark to Theorem 2 the minimal value of the K^* is in the cases (i) and (iii) separated into two independent parts of minimum cost for special problems with setup numbers to be equal one.

(i) The minimum cost is equal the sum of the independent minimum cost $R = \sqrt{2dr\beta^2(h-u)}$ of repairing and storing the fraction $\beta x_R, x_R = \sqrt{2dr/(h-u)}$, with the holding cost input $(h-u)$ and of the minimum cost $P = \sqrt{2ds(\alpha^2 h + \beta u(1+\beta))}$ of producing and storing the fraction $\alpha x(\alpha)$ of new products and storing also the used products. As it can be seen the holding cost u subtracted in the first expression is then added in the second one, and due to (14) $\beta x(\alpha) \geq x_R$. Since the function P is convex, this is true also for $K_1(\alpha)$.

(iii) The minimum cost is equal the sum of the independent minimum cost $R = \sqrt{2dr\beta(h\beta + u)}$ of repairing and storing the fraction $\beta x(\alpha)$ and the minimum cost

$P = \sqrt{2dsh\alpha^2}$ of producing and storing the fraction αx_P, $x_P = \sqrt{2ds/h}$. Obviously, the relation $\alpha x(\alpha) \geq x_P$ holds. Now, the function R is concave and so does $K_3(\alpha)$.

It follows from those properties, that the marginal minimum cost as a function of α increases up to α_2 and falls behind. In other words, the marginal cost increases with decreasing degree of weak preference of repair to disposal, and decreases with increasing preference of disposal to repair.

3. Comparison of mixed and extreme waste disposal rates

Now the convexity of the cost function $K_2(a)$ is assumed.

Lemma 2. *The relation* $g(\alpha, e) \geq \min\{K_3(0) + dk, K_3(1) + d(b+e)\}$ *for* $0 \leq \alpha \leq 1$ *is fulfilled.*

Proof. The function $g(\alpha, e)$ is convex for $0 \leq \alpha \leq \alpha_2$ and coincides at $\alpha \geq \alpha_2$ with the function $K_3(\alpha)+R(\alpha, e)$, which is concave on the whole interval [0, 1]. Then, due to the continuous differentiability of $K(\alpha)$ and $g(\alpha, e)$, the relation $g(\alpha, e) \geq K_3(\alpha) + R(\alpha, e)$ for $0 < \alpha < 1$ holds and $g(\alpha, e)$ is bounded from below by the minimum of $K_3(0)+ dk$ and $K_3(1)+d(b+e)$. □

Theorem 3. *One of the pure strategies of total repair (no waste disposal) or of total disposal (no repair) is optimal, i. e.* $g(\alpha, e) \geq min\{g(0, e)\ g(1, e)\}$ *is true.*

Proof. The analysis of the function $K_3(\alpha)$ for $\alpha = 0 \vee 1$ shows that $K_3(0) = K_0 = \sqrt{2dr(h+u)}$ and that $K_3(1) = K_1 = \sqrt{2dsh}$. Then it follows from the previous Lemma that no mixed strategy can be better than one of the pure solutions. □

Remark. In the case of linear holding, repair, production and waste disposal cost and of free choice of the waste disposal rate between 0 and 1 one of the pure strategies to repair or to dispose of all used products is optimal. Probably these pure strategies are technologically not feasible and there will always exist some unrepairable used products which are to be disposed of. In this case mixed strategies for the problems [ECOL→ECON] and [ECON→ECOL] seem to have practical relevance.

The optimal pure strategy can be simply found by comparing the values of $K_0 + dk$ and $K_1 + d(b+e)$: Repair is (strongly) preferred to disposal, if and only $\sqrt{r(h+u)} < (e+b-k)\sqrt{d/2} + \sqrt{sh}$ holds. If the waste disposal cost e drops down the preference shifts to the disposal option.

In the example considered in the section 2.2. $g(0, e) = 168.322$ and $g(1, e) = 218.322$ hold, while the minimal value for mixed strategies is above 200. Thus, the repair option would be preferred in the example.

4. The changes of setup numbers, lot sizes and collection intervals

If $u > 0$ and the formulae from Theorem 2 are combined the following results can be easily obtained for the repair and manufacturing lot sizes.

Lemma 3. *The repair setup number fulfills* $m(\alpha) \leq max\left(1, \sqrt{\frac{s(h-u)}{ru}}\right)$, *i.e. it is bounded by a constant, while the production setup number becomes infinitely large with increasing waste disposal rate, i. e.*

$$n(\alpha) \geq \frac{\alpha}{\sqrt{1-\alpha}}\sqrt{\frac{rh}{s(h+u)}}.$$

Remark. The setup numbers show conflicting behavior: The changes in the relation *s/r* affects the bounds differently, i. e. one bound will rise and the other will fall, while changes in *h/u* produce the same effects for both the bounds.

The lot size $x(\alpha)$ and the length $T(\alpha)$ of the collection interval behave similarly: While $x(\alpha)$ has obviously an upper bound for $\alpha \leq \alpha_1$, it is unbounded for

$$\alpha \geq \alpha_2:\ x(\alpha) \geq \sqrt{\frac{2dr}{(1-\alpha)(h+u)}},$$

where the lower bound might become infinitely large. Because of $x(\alpha)/d = T(\alpha)$ the same relations are true for the length of the collection interval.

In the model, however, not the lot sizes $x(\alpha)$ are used, but a certain number of repair lots and a certain number of manufacturing lots. Their size is estimated by the following

Lemma 4. (i) *Let* $\alpha \leq \alpha_1$. *Then* $\beta x(\alpha)$ *units of used products are repaired in* $m(\alpha)$ *lots of the size* $\sqrt{2rd/(h-u)}$ *and one lot of size* $\alpha x(\alpha) \leq \sqrt{2ds/h}$ *is newly manufactured.*

(ii) *If* $a \geq \alpha_2$ *then* $ax(\alpha)$ *units of new products are manufactured in* $n(\alpha)$ lots of size $\sqrt{2sd/h}$ *and one lot of size* $\beta x(\alpha) \leq \sqrt{2dr/(h-u)}$ *is repaired.*

The sizes of all repair and manufacturing lots are bounded, and only their number might change and even become infinitely large. Therefore in the case (i), when many lots of used products are repaired and only one lot is produced newly and the appropriate quantity is disposed of, it makes sense to speak on a preference of repair to disposal, and in the case (iii), when the opposite situation occurs, to speak on the opposite preference relation. For a given α it is not clear, if it is preferable or not. Perhaps, one wants it to be as small as possible. The discussed relation of repair lots to production lots gives some new insight, and at the same time a new (cost oriented) criterion for estimating the relation between repair and disposal.

Finally, it should be noted that, if the waste disposal rate tends to one, some discontinuous situation occurs: The infinitely large collection interval $T(\alpha)$ and the infinitely large manufacturing setup number $n(\alpha)$ collapse to the values of the traditional model $T(1) = \sqrt{2s/dh}$ and $n(1) = 1$, $m(1) = 0$.

5. Conclusion

The main result of the paper on the optimality of the pure strategies of total disposal or total repair is also true, if the lot size number are integers. Although the objective function is then not convex-concave as in the continuous case, the function $g(\alpha, e)$ is a lower bound for the integral function $g_I(\alpha, e)$ and hence Theorem 3 holds again. The integral problem is, however, not trivial and it will be subject of another study.

Acknowledgement. The author thanks Dr. Imre Dobos for many discussions and suggestions in solving the auxiliary problem in this paper and two reviewers for valuable comments.

References

1. Betz St (1995) Kostenorientierte Planung der Entsorgungslogistik, Z Planung, 321–339
2. Bogaschewski R (1995) Natürliche Umwelt und Produktion. Gabler, Wiesbaden
3. Dinkelbach W, Rosenberg O (1994) Erfolgs- und umweltorientierte Produktionstheorie. Springer-Verlag, Berlin
4. Dyckhoff H (1992) Betriebliche Produktion – Theoretische Grundlagen einer umweltorientierten Produktionswirtschaft. Springer-Verlag, Berlin
5. Gupta MC (1995) Environmental management and its impact on the operations function, Int J Oper & Dec Manag 15(8): 34–51
6. Haasis H-D (1994) Planung und Steuerung emissionsarm zu betreibender industrieller Produktionssysteme. Physica-Verlag, Heidelberg
7. Kelle P, Silver EA (1989) Purchasing policy of new containers considering the random returns of previously issued containers, IIE Transactions 21(4):349–354
8. Laan EA van der, Dekker R, Ridder AAN, Salomon M (1994) An (s,Q) inventory model with remanufacturing and disposal, erscheint in: Int J Produc Econ, to appear
9. Laan EA van der, Salomon M, Dekker R (1995) Production planning and inventory control for remanufacturable durable products, Report 9531/A, Erasmus University Rotterdam
10. Richter K (1994) An EOQ repair and waste disposal model, Proceedings of the Eight International Working Seminar on Production Economics, Vol. 3, 83–91, Igls/Innsbruck
11. Richter K (1996) The extended EOQ repair and waste disposal model. Int J Produc Econ 45:443–447
12. Richter K (1996) The EOQ repair and waste disposal model with variable setup numbers. Eur J Oper Res 96:313–324
13. Richter K (1996) Modellierung von kombinierten Wiederverwendungs- und Entsorgungsprozessen. In: Wildemann H (eds): Produktions- und Zuliefernetzwerke. TCW Transfer-Centrum-Verlag, München, pp 279–291
14. Steven M (1994) Produktion und Umweltschutz. Gabler-Verlag, Wiesbaden
15. Thierry MC, Salomon M, Nunen JAEE van, Wassenhove LN van (1995) Strategic production and operations management issues in product recovery management, Calif Manag Rev 37(2): 114–135