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Stability of the constant cost dynamic lot size model

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Abstract: A mathematical analysis of the dynamic lot size model with constant cost parameters is provided. First, stability regions for so called generalized and optimal solutions are found, which show how the cost input may vary, leaving the solution valid. Based on these results a BASIC dialog program has been designed to display the optimal solution and the stability regions to the decision maker. Secondly, an estimation of the initial optimal solution is given for the case, when the cost inputs leave the stability region.

Keywords: Inventory, dynamic programming, parametric programming

1. Introduction

A mathematical analysis of the dynamic lot size model with constant cost parameters is provided in this paper. Stability regions for generalized and optimal solutions are found, which show how cost inputs may vary, leaving the solution valid. The generalized solutions from Wagner–Whitin's algorithm are used to generate an optimal solution, and the corresponding stability region can be applied to estimate whether, with respect to the inaccuracy of the cost inputs of the model, the optimal solution will suit the decision maker's individual idea of the robustness of a solution.

There is no one-to-one correspondence between generalized and optimal solutions. Therefore, the neighbouring stability regions, which may also cover the optimal solution, have to be studied. Based on these results a BASIC dialog program has been designed to perform the calculations on the G.D.R. A 5110 personal computer. The program can be used to list all possible optimal solutions for a given demand structure.

The dynamic lot size model is one of the best known standard models in Operations Research, and it has been studied by several authors [1,3,6–9]. Generalizations of the model in multistage processes and in multi-item systems have also been investigated [4,5,9]. In all these models the optimal solutions provide lot sizes which minimize the sum of fixed costs and of holding costs. In practice, however, one wants to know more than only an optimal solution. As in linear programming [2], it is of great interest (i) to find sets of inputs leaving an optimal solution valid and (ii) to estimate the value of an optimal solution if the inputs change very much.

Since the cost inputs in dynamic lot size models are highly inaccurate, a stability analysis for these inputs will be provided in the paper. Some formulae will be given, which display the exact region of the cost parameters for generalized solutions generated by Wagner–Whitin's algorithm, and an estimate of this solution for the case will be found when the changing cost inputs do not belong to the stability region. Before discussing the main results and their application, the dynamic lot size model will be formulated.

2. The dynamic lot size problem

Let $c > 0$ denote the set-up cost and let $h > 0$ denote the per unit per period holding cost of

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some item. Then the production figures $x_t \geqslant 0$ have to be chosen such that the given deterministic demand $d_t \geqslant 0$ of this item is satisfied in all periods $t = 1, 2, \ldots, T$. If the stock at the end of the period t is denoted by y_t, then the demand is satisfied if y_t is nonnegative. Usually the assumption is made that the stock equals zero at the beginning and at the end of the considered planning period, i.e. $y_0 = y_T = 0$. Then the problem of minimizing the total costs can be expressed by the following model:

$$c \sum_{t=1}^{T} \operatorname{sign} x_t + h \sum_{t=1}^{T} y_t \to \min, \tag{1}$$

$$y_t = y_{t-1} + x_t - d_t, \tag{2}$$

$$x_t \geqslant 0, \quad y_t \geqslant 0, \qquad t = 1, 2, \ldots, T, \tag{3}$$

$$y_0 = y_T = 0. \tag{4}$$

Let us consider a simple example to illustrate the results which the paper tries to obtain. If $T = 3$, $c = 5$, $h = 2$, $d_1 = 3$, $d_2 = 2$, $d_3 = 1$, then model (1)–(4) is given by the subsequent relations:

$$5(\operatorname{sign} x_1 + \operatorname{sign} x_2 + \operatorname{sign} x_3) + 2(y_1 + y_2) \to \min,$$

$$y_1 = x_1 - 3, \quad y_2 = y_1 + x_2 - 2, \quad 0 = y_2 + x_3 - 1,$$

$$x_1, x_2, x_3, y_1, y_2 \geqslant 0.$$

Then $x_1 = x_2 = 3$, $x_3 = 0$, $y_1 = y_3 = 0$, $y_2 = 1$ provides the unique optimal solution which can be shown to be valid if the ratio c/h belongs to the interval [1,3]. If $c/h > 3$ then $x_1 = 6$, $x_2 = x_3 = 0$, $y_1 = 3$, $y_2 = 1$, $y_3 = 0$ and if $c/h < 1$, then $x_1 = 3$, $x_2 = 2$, $x_3 = 1$, $y_1 = y_2 = y_3 = 0$ are optimal solutions, respectively. If, on the one hand, the decision maker is sure that the relation c/h of fixed and holding costs will remain within this interval, he can use the generated solution. On the other hand, if this relation is changing essentially, it will be of great interest to know, whether the ratio

$$\text{costs of the initial optimal solution/costs of the solution optimal with respect to the new costs} \tag{5}$$

is significantly greater than one or not. It can be shown that, if in the above example c/h becomes greater than three, then this ratio is bounded by $c/3h$, i.e. if, for instance, $c = 5$ and $h = 1$ then relation (5) will be bounded by 5/3. The decision maker has to decide then whether the initial solution can be used or if he must turn to the current optimal solution for $(c, h) = (5, 1)$.

Actually, there are also other reasons for investigating the stability of such solutions. The algorithms developed for the extensions of Wagner–Whitin's model (1)–(4) employ in some cases the idea of decomposition [4,5] and of Pareto-optimality [6], and models of this type have to be solved many time with slightly modified inputs. Then, with the knowledge of the structure of the stability regions, one might be able to speed up those algorithms.

3. Generalized solutions of Wagner / Whitin's model

Because of the positivity of the cost inputs, it can be shown that $d_t = 0$ implies $x_t = 0$ for all t and all optimal solutions. Then d_1 and d_T can be assumed to be positive. Let $d(k, l)$ denote the demand for the periods $k+1, \ldots, l$, i.e.

$$d(k, l) = \sum_{t=k+1}^{l} d_t.$$

Then $y_{t-1} > 0$ implies that $x_t = 0$ and $y_{t-1} = 0$ implies that $x_t \in \{d(t-1, t), \ldots, d(t-1, T)\}$. This property has been proved by many authors for at least one optimal solution. Here it holds for all optimal solutions. Let the following symbols

$$h(k, l) = h \sum_{t=k+1}^{l} (t-k-1) d_t$$

and

$$c(k, l) = c + h(k, l)$$

for $k < l$ and $d_{k+1} > 0$ express the total costs over the periods $k+1, \ldots, l$ if $y_k = y_l = 0$.

The set $T_0 = \{t: d_t > 0\}$ covers all periods with positive demand. Then the minimal total costs f_T can be found by the following procedure (Wagner/Whitin's algorithm):

$$f_0 = 0, \; f_1 = c, \quad \text{for } t \geqslant 2 \text{ and } t \in (T_0 - \{1\}) \cup \{T\}, \tag{6}$$

$$f_t = \min\{c(k, t) + f_k \colon k \leqslant t-1, \; k \in T_0 - \{1\}\}.$$

Procedure (6) applied to the example leads to

$$f_0 = 0, \qquad f_1 = 5,$$

$$f_2 = \min\{c(0, 2) + f_0, \; c(1, 2) + f_1\}$$

$$= \min\{9 + 0, \; 5 + 5\} = 9,$$

$$f_3 = \min\{ c(0, 3) + f_0, c(1, 3) + f_1, c(2, 3) + f_2 \}$$
$$= \min\{13 + 0, 7 + 5, 5 + 9\} = 12,$$

i.e. the minimal value equals 12.

Let $f(k, t) = c(k, t) + f_k$ and let the parameters $k(t)$ be introduced by

$$f_t = f(k(t), t) \tag{7}$$

for suitable t. Then an optimal solution can be found using these parameters.

Algorithm

1. $t := T$.
2. $x_{k(t)+1} := d(k(t), t)$, $x_k := 0$ for $k = k(t) + 2, \dots, t$.
3. If $k(t) = 0$ stop, else $t := k(t)$ and go to step 2. Output $x_1, x_2, \dots, x_T$.

The components of vector y can be found by formula (2).

It can be easily seen that in the example $k(1) = k(2) = 0$ and $k(3) = 1$. Then using these data the algorithm can be started:

1. $t = 3 \Rightarrow$ 2. $x_2 = d(1, 3) = 3$, $x_3 = 0$.
2. $k(3) \neq 0 \Rightarrow t = k(3) = 1 \Rightarrow$ 2. $x_1 = d(0, 1) = 3$.
3. $k(1) = 0$, stop.

The values $y_1 = 0$, $y_2 = 1$ and $y_3 = 0$ can be determined by (2).

The parameters as well as the optimal solution are usually not unique. The collection

$$K = \{ k(t) \}_{t \in (T_0 - \{1\}) \cup \{T\}}$$

will be called a generalized solution. It should be noted that a generalized solution contains much more information than an optimal solution. On the one hand, it follows from formula (6) that K can be used to determine optimal solutions for all problems (1)–(4) with a number of periods $T' \leqslant T$, where $T' \in (T_0 - \{1\}) \cup \{T\}$.

On the other hand, the optimal solution found can be also derived from the generalized solution $\{0, 1, 1\}$. Because of this property one can expect that the stability region of a generalized solution will be only a subset of the stability region of the corresponding optimal solution.

The main result needed in the next section is the following: Every generalized solution fulfils the inequality

$$k(l) \leqslant k(t) \tag{8}$$

for all suitable $l < t$.

It follows from this inequality that the parameters $k(t)$ found by (6) and (7) can be determined by comparing only the values $f(k, t)$ for $k = k(l), k(l) + 1, \dots, t - 1$. This technical result will help to establish the exact stability region.

4. Stability of generalized solutions

Let K be a generalized solution generated by (6) and (7). For which cost inputs except (c, h) is K valid? In order to study this problem the parameters f_t', $f'(k, l)$, $c'(k, l)$, $h'(k, l)$ etc. will be introduced for cost inputs (c', h') with the same meaning as f_t, $f(k, l)$, $c(k, l)$, $h(k, l)$ for (c, h).

The number of set-ups $c(t)$ for all suitable t will be found by the formulae

$$c(0) := 0 \quad \text{and} \quad c(t) := c(k(t)) + 1. \tag{9}$$

For example, the values $c(0) = 0$, $c(1) = c(2) = 1$ and $c(3) = 2$, i.e. if the three-period problem is considered, then the production figures are twice positive.

Using property (8), the subsequent assertion can be proved easily.

Lemma 1. *Let K be a generalized solution. Then the inequality*

$$c(l) \leqslant c(t)$$

holds for all suitable $l < t$.

Now let the parameters

$$r(k, t) = (f(k, t) - f_t)/(c(k(t)) - c(k)) \tag{10}$$

be introduced for all $k < t$ with $c(k(t)) \neq c(k)$ and $k, t \in (T_0 - \{1\}) \cup \{T\}$.

In the example $r(1, 2) = (f(1, 2) - f_2)/(c(0) - c(1)) = (10 - 9)/(0 - 1) = -1$ and $r(0, 3) = 1$ can be found.

Then the general bounds

$$\text{low} = \max_t \max_k \{ r(k, t) \colon c(k) > c(k(t)) \}, \text{ and}$$
$$\text{up} = \min_t \min_k \{ r(k, t) \colon c(k) < c(k(t)) \} \tag{11}$$

can be determined where low and up can be set equal minus and plus infinity, if the values are not defined. In the example the bounds are equal low $= -1$ and up $= 1$.

Theorem 1. *Let $h' = h$ and let $c' > 0$ be such positive number that $c' \in [c + \text{low}, c + \text{up}]$. Then the generalized solution is valid for (c', h') and $f_t' = f_t + c(t)(c' - c)$ holds for all suitable t.*

Proof. By induction. Since $d_1 > 0$, by assumption $k(1) = 0$, $c(1) = 1$, $f_1' = f_1 + c(1)(c' - c)$ and $k(1)$ is valid for all $c' > 0$ of the one-period problem.

Now let the general case t be studied under the usual assumption of induction. If $f'(k, t) < f(k(t), t)$ for some $k \neq k(t)$ then $f_k' + c'(k, t) < f'_{k(t)} + c'(k(t), t)$ holds. It follows from the assumption of induction and from $c'(k, t) = c(k, t) + (c' - c)$ that $f(k, t) + (c(k) + 1)(c' - c) < f_t + (c(k(t)) + 1)(c' - c)$. If $c(k(t)) = c(k)$, then the inequality reduces to $f(k, t) < f_t$, which contradicts the procedure (6). Let $k < k(t)$. Then it follows from Lemma 1 that $c(k(t)) \geqslant c(k)$ is fulfilled. If this inequality is strict, then the above relation can be rewritten as

$$(f(k, t) - f_t)/(c(k(t)) - c(k)) < c' - c$$

and there is a contradiction to the choice of c', i.e. $c + \text{up} < c'$. The other case, if $k > k(t)$, can be studied in the same way.

Thus $k(t)$ is part of the generalized solution for (c', h') and

$$f_t' = f'_{k(t)} + c'(k(t), t) = f_t + (c(k(t)) + 1)(c' - c)$$
$$= f_t + c(t)(c' - c). \qquad \square$$

It follows from this statement that $K = \{0, 0, 1\}$ is the generalized solution for all pairs (c', h') with $h' = 2$ and $c' \in [4,6]$.

The next theorem will show that the parameters $c + \text{low}$ and $c + \text{up}$ do not depend on c but on K being valid.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled and let $r'(k, t)$, low$'$ and up$'$ be the parameters (10) and (11) for (c', h'). Then $r'(k, t) = r(k, t) - c' + c$ and*

$$c + \text{up} = c' + \text{up}', \qquad c + \text{low} = c' + \text{low}'$$

hold.

Proof. Using the result of Theorem 1 one can easily see that

$$r'(k, t) = (f(k, t) - f_t + (c(k) - c(k(t)))(c' - c))/(c(k(t)) - c(k))$$
$$= r(k, t) - c' + c.$$

The other relations follow immediately from the above equality. $\square$

In order to show that $[c + \text{low}, c + \text{up}]$ provides the exact stability region for the generalized solution, the subsequent statement is useful.

Theorem 3. *Let K be the generalized solution for (c, h) and let $h' = h$ but $c' < c + \text{low}$ or $c' > c + \text{up}$. Then K is not valid for (c', h').*

Proof. Let K be valid and let $c' < c + \text{low}$. Then $f_t' = f_t + c(t)(c' - c)$ for all t as in Theorem 1. Now let $r(k, t) = \text{low} > c' - c$, i.e.

$$(f(k, t) - f_t)/(c(k(t)) - c(k)) > (c' - c)$$

and

$$f_t + (c(k(t)) + 1)(c' - c)$$
$$> f(k, t) + (c(k) + 1)(c' - c).$$

Thus $f_t' > f'(k, t)$. Contradiction. The case $c' > c + \text{up}$ can be handled similarily. $\square$

Now let the stability region of the generalized solution, i.e. the set of all cost inputs for which this solution is valid, be expressed more conveniently.

Theorem 4. *Let K be a generalized solution for the inputs (c, h). Then it is valid for all c', $h' > 0$ satisfying*

$$(c + \text{low})/h \leqslant c'/h' \leqslant (c + \text{up})/h$$

and only for those inputs.

Proof. If K is valid for some (c', h') then it is valid for $(c', h')/(h'/h) = (c'h/h', h)$ and it follows from Theorem 1 that $c + \text{low} \leqslant c'h/h' \leqslant c + \text{up}$. On the other hand, let (c', h') satisfy the mentioned inequality. Then $(c'h/h', h)$ provides inputs for which K is valid, i.e. K is valid for (c', h'). The generalized solution is valid for only these inputs since Theorem 3 holds. $\square$

According to this theorem the stability region of $\{0, 0, 1\}$ can be expressed by $2 \leqslant c'/h' \leqslant 3$.

Corollary. *The stability region of a generalized solution is a convex cone in* $\mathbb{R}^2_+$. *If the demand vector* $(d_1, d_2, \ldots, d_T)$ *is given, then there is a finite number of stability regions covering* $\mathbb{R}^2_+$.

If, for instance, $d_1 = 3$, $d_2 = 2$, $d_3 = 1$ from the example is considered, then the following stability regions and associated generalized solutions can be found:

(i) $0 < c/h \leqslant 1$ for $\{0, 1, 2\}$,
(ii) $1 \leqslant c/h \leqslant 2$ for $\{0, 1, 1\}$,
(iii) $2 \leqslant c/h \leqslant 3$ for $\{0, 0, 1\}$,
(iv) $3 \leqslant c/h$ for $\{0, 0, 0\}$.

A BASIC dialog program has been designed for the A 5110 G.D.R. personal computer to perform procedure (6), the algorithm and the stability analysis according to Theorem 4. By this program the neighbouring stability regions can be also checked as to they are covered by the determined optimal solution or not. In the example the generalized solution associated with the stability region (ii) generates the same optimal solution as it was found for $c = 5$ and $h = 2$. Therefore the real stability region of the optimal solution is given by $1 \leqslant c/h \leqslant 3$.

5. Approximation of the optimal solution

If the cost inputs change within the stability region of the optimal solution, no correction is necessary and the solution can be used. It is, of course, of interest to know whether one can continue to apply the known solution. This is the case, if the initial optimal has a value which does not differ essentially from that of the new optimal solution which, however, may be unknown to the decision maker. Although he is able to answer the question quickly by applying the dialog program one must ask for simpler formula in decision making.

Therefore, let K be a generalized solution found for the cost inputs (c, h). The problem is how one can estimate the solution if the inputs change to (c', h') and this pair does not belong to the stability region. In order to study this problem, let f' denote the minimal costs for (c', h') and let $f'(c, h)$ denote the costs arising with respect to (c', h') if K is used. The ratio $f'(c, h)/f'$ will then tell if the previous solution is sufficiently 'good'.

Let two multipliers be introduced which use the parameters low and up from the stability analysis from Section 4.

$$m = c'h/(h'(c + \text{up})),$$

$$n = h'(c + \text{low})/(c'h).$$

If low or up are not finite, then the corresponding multiplier will be set equal to zero. Without proof the following theorem will be stated:

Theorem 5. *Let* $e = \max(m, n)$. *Then* $f'(c, h) \leqslant ef'$ *or* $f'(c, h)/f' \leqslant e$ *holds.*

Let the same example be studied again. For $c = 5$ and $h = 2$ $m = c'/(3h')$ and $n = 2h'/c'$. If $c'/h' > 3$ then $m > 1$ and $f'(5, 2)/f' \leqslant c'/(3h')$. This bound shows that the value of the initial solution is not greater than mf', and considering that $f'(5, 2) = 2c' + h'$ and that $f' = c' + 4h'$ one can see that $(2c' + h')/(c' + 4h') \leqslant c'/(3h')$ for $c' \geqslant 3h'$.

Let $c' = 8$ and $h' = 2$. Then $f'(5, 2)/f' = 9/8 < e = m = 4/3$, i.e. the bound is not sharp. Taking into consideration that the value of e can be greater than the real value of the ratio (5), the decision maker has to decide if e is sufficiently small or not.

References

[1] Bitran, G., Magnant, T., and Yanasse, H., "Approximation methods for the uncapacitated dynamic lot size problem", *Management Science* 30 (1984), 1121–1140.

[2] Dantzig, G.B., *Linear Programming and Extensions*, Princeton University Press, 1963.

[3] Florian, M., Lenstra, J.K., and Rinnooy Kan, A.H.G., "Deterministic production planning: Algorithms and complexity", *Management Science* 36 (7) (1980) 669–679.

[4] Kalymon, B.A., "A decomposition algorithm for arborescence inventory systems", *Operations Research* 20 (4) (1974) 860–874.

[5] Richter, K., "*Dynamic Problems of Discrete Programming*, Akademie, Berlin, 1982 (German), Radio i svjas Moscow 1985 (Russian)

[6] Richter, K., "The two-criterial dynamic lot size problem", *System Analysis, Modelling, Simulation* 3 (1) (1986) 99–105.

[7] Wagner, H.M., *Principles of Operations Research*, Prentice-Hall, Englewood Cliffs, NJ, 1969.

[8] Wagner, H.M., and Whitin, T.M., "Dynamic version of the economic lot size model", *Management Science* 5 (1) (1985), 89–96.

[9] Zangwill, W.I., "A deterministic multiproduct multi-facility production and inventory model", *Operations Research* 14 (3) (1966) 486–507.