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# A Theory of Coalition Bargaining

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# A Theory of Coalitional Bargaining in Democratic Institutions

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**Abstract**

When voting takes place in democratic institutions, we find (either explicitly or implicitly) that there is an agenda setter or a formateur. Such players are uniquely able to make substantive proposals for given topics. Their statuses remain intact even after rejections of proposals, but they must revise rejected proposals constructively (e.g. towards a compromise). We model this in a general environment, show that the equilibrium outcome is generically unique, and characterize it explicitly. The equilibrium outcome is robust to (partially) binding communication between the formateur and the voters. As illustrations, we consider majority bargaining about a cake (leaned on Baron and Ferejohn, 1989), where the formateur ends up being a perfect dictator, and a model of legislative voting (leaned on Jackson and Moselle, 2002), where he is a dictator if his ideological position is within the quartiles of the parliament. In these cases, our model implements (reversed) McKelvey majority paths. Depending on the valuations, the formateur's power may be weakened when parliamentary decisions can be revised, as this may facilitate tacit collusion amongst the voters.

*JEL–Codes:* C72, C78, D72
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# 1 Introduction

When majorities are sufficient to implement decisions, then coalitions form; either implicitly, by players that vote in the same way, or explicitly, by players that sign coalition contracts. Models of coalitional bargaining help us to understand which decisions are made and which coalitions form. In this paper, we define a general framework to analyze coalitional bargaining games and we characterize the perfect equilibrium outcomes for bargaining rules that apply to a wide range of democratic voting institutions. We prove existence of equilibria, uniqueness of the outcomes, and provide a unified characterization of the outcomes. Based on that, specific assumptions concerning the option sets, the preference functions, and the voting rules allow us to derive specific predictions.

Our model has the following three cornerstones. First, the formateur remains in his position even after rejections of proposals (the formateur is the player with the right to make proposals that would be voted upon). Secondly, when the formateur revises a previously rejected proposal, then he has to do so in a constructive way (to avoid "blame game" politics, as discussed below). Thirdly, there *may* be communication between the formateur and the other players that differs from cheap talk (e.g. when misleading communication can be retaliated within existing hierarchies). These cornerstones have predecessors in the literature, but typically, they have been studied in isolation and in restrictive circumstances. Our analysis generalizes these studies and unifies the respective classes of model.

The assumption that formateurs stay in their positions even after rejections applies to both major strings of political bargaining: government formation and parliamentary legislation. It applies to government formation when the head of state appoints the formateur in the beginning of the bargaining phase (as in the Netherlands, Belgium, Finland). The formateur would then ask the other players to sign coalition contracts, to partially fix the portfolio allocation and the political programme (Cheibub et al., 2004). Coalition contracts are offered behind the scenes, and thus, the formateur remains in his position even if such an offer should be rejected.

In the second case (parliamentary legislation), there typically are unique players with the ability to make substantive proposals (agenda setters). Their time horizons are large, as proposals can be revised quickly and as they can be made informally (e.g. at a cabinet table), and thus agenda setters are formateurs in our sense. Their proposal ranges may be restricted (thematically), which we take as given. Depending on the context, agenda setters can be ministers (Laver and Shepsle, 1990), presidents (Persson et al., 1997, 2000, and Primo, 2002), chairmen or board members within parties (Cox and McCubbins, 1993), or majority parties (coalitions) in parliaments (e.g. in the UK and in Ireland, Döring, 1995, in the US House of Representatives, Cox and McCubbins, 2005, and in the German Bundestag, Loewenberg, 2003). Apparently, the required voting shares depend on the context of the game, but as we show, there is a common intuition behind the bargaining games that is independent of the voting rules.

By assuming that the formateur is not replaced after rejections of proposals, we depart from the branch of literature that followed Baron and Ferejohn (1989); for instance Baron (1991), Chatterjee et al. (1993), Okada (1996), Seidmann and Winter (1998), and Ray and Vohra (1999). This literature describes coalitional bargaining in non–institutional circumstances, e.g. cartel formation and government formation when the Head of State does not appoint a formateur before a coalition has formed (as in Italy). In another related branch of models, it is assumed that the formateur commits to a coalition before any distributive or programmatic aspects are negotiated (in particular, in Diermeier et al., 2002, 2003, but see also Bloch, 1996, and Baron and Diermeier, 2001). Our model generalizes this branch in that the formateur can require potential coalition partners to commit to some distributive and programmatic aspects (via coalition contracts) before he commits to a coalition. The results of the multilateral negotiation concerning any remaining aspects are taken as given in our model.

Our second main assumption is that rejected proposals have to be revised constructively. This is related to the literature on blame game politics (Rohde and Simon, 1985; Smith, 1988; Woolley, 1991). A formateur who is unwilling to compromise can be blamed as being "unconstructive" by the players that have to vote on his proposals, and following such blames, his reputation may suffer significantly (see Groseclose and McCarty, 2001). In turn, a formateur who revises a rejected proposal towards a compromise can not be blamed of being unconstructive. Additionally, we say that formateurs can not be blamed of being unconstructive when they come up with entirely new proposals. Implicitly, we show that the equilibrium outcome does not depend on whether the latter applies or not. To simplify the notation, we assume that the formateur would have to leave office after his *first* unconstructive proposal, i.e. after his first proposal that is no compromise and not new (but any other finite limit leads to the same equilibrium outcome). Then, an outside option would apply. The resulting bargaining model, where the formateur can revise rejected proposals but must be constructive, is called *constructive proposals game*. We show that the solution is generically unique under perfectness and characterize it.

The model of constructive proposals does not allow for communication between the formateur and other players. In particular, the formateur could be informed by the voters about the proposals they would accept (or, he could ask them). Such communication becomes relevant if it is not cheap talk. As it appears, it is not cheap talk in our circumstances. For, the structures of parties and coalitions are hierarchical and resemble legal partnerships (Cox and McCubbins, 1993). As a result, formateurs would be able to observe misleading communication, and they have a number of measures to retaliate it: lack of promotion, demotion, and expulsion from the coalition (Cox and McCubbins, 1994). To investigate the robustness of our predictions to such communication, we first analyze a case of perfectly binding communication: when players signaled that they would accept a given proposal, then they would indeed accept it. In the corresponding *sealed offers game*, the formateur asks the players which options they would accept, and of the emerging possibilities, he chooses the one that he prefers most. We prove outcome equivalence to the non–communication model, and further below, we extend this equivalence to a game where communication is not binding, but misleading communication is associated with positive costs.

The predictions of our model are illustrated in Section 3. On one hand, we consider a parliament that decides on how to allocate a cake (following Baron and Ferejohn, 1989). On the other hand, we derive the decision when there is both, a distributive dimension (cake allocation) and an ideological dimension (a political programme is chosen from the real line). The second model follows Jackson and Moselle (2002). In any of the respective equilibria, the formateur approximately gets the whole cake. Furthermore, in the Jackson–Moselle model, he can implement his ideal programme if his position is within the quartiles of the parliament, and otherwise he can at least implement the programme at the preferred quartile. This does not depend on the "size" of the cake to be allocated. Thus, the formateurs get close to become dictators in these model families, and as we illustrate below, our model provides an implementation of McKelvey (1976, 1979) majority paths in these cases. In general, however, the formateur is not able to implement his most favorite option; an example is given below.

Given the existing literature, these results are somewhat surprising. First, median voters are irrelevant in the considered cases; limiting members of the parliaments are quartile voters (if at all). Secondly, interpreting the sealed offers game, one may be surprised that the voters would reply (when asked) that they accept proposals as extreme as the above ones; but let us note that our results require a high order of iterative reasoning. Thirdly, the extremity is surprising even if we concentrate on the constructive proposals game. Clearly, there is only one player who can make proposals, but this does generally not imply uniqueness of the equilibrium outcomes (least of all in discrete option spaces). Additionally, the formateur can not make take–it–or–leave–it offers, as in one–round games, which suggests that his bargaining power is less here than in one–round games. His bargaining power appears to be weakened further as he must be constructive—if the voters reject a given proposal, they know that a new proposal or a compromise proposal would follow. As a result, if the voters are sufficiently patient, then they could wait until the formateur proposes an appropriate compromise. Regardless of their patience, however, they do not wait; they compete for pieces of the cake.

In Section 4, we examine a variation of this model where previous parliamentary decisions are canceled when new decisions are made (this is loosely related to Dixit et al., 2000). Here, the possibility of revisions is acknowledged by the voters when they evaluate the implications of supporting a given proposals. This may lead to "tacit collusion" amongst the voters. They do not generally compete anymore, but they can induce that decisions may trigger later revisions, following which all players would be worse off. As a result, they would not support extreme proposals anymore. There are several further explanations for less extreme predictions than this one, and we discuss some of them in the concluding Section 5. A formal analysis of the case with revisable decisions and some proofs are relegated to the appendix.

## 2 Sealed Offers vs. Constructive Proposals

The exposition of the results is simplest if we start with defining and analyzing the sealed offers game. We then show that the sealed offers game is outcome equivalent to a "descending proposals" game. This game is dynamic and the formateur proposes the options in descending order (based on his preferences) until an option is accepted. We say that two games are *outcome equivalent* if the sets of outcomes sustained in equilibrium are equivalent. In our cases, these sets will be singletons. At the end of this section, we extend the outcome equivalence to the constructive proposals game, which is a generalization of the descending proposals game. Given the nature of these games, we find it most convenient to define them on discrete option spaces. For instance, if the players make offers to the formateur, then we say that they choose a set of offers from a discrete set. Likewise, discreteness simplifies the definition of "constructive revisions" of proposals (without discreteness, some parameters would be required). Finally, the solution concepts would have to be significantly more complex in continuous games. Nonetheless, it appears that the basic intuition behind our arguments extends to games defined on continuous option spaces.

Technically, the sealed offers game resembles sealed–bid auctions, and in particular, it resembles menu auctions (Bernheim and Whinston, 1986) and team selection (Bolle, 1995). In menu auctions, the players can contingent their bids on the eventual object allocation. In the sealed offers game that we analyze, the valuation functions are more general, and if objects were to be auctioned off, the bids could be contingent not only on the resulting allocation but also on the winning bids. Because of the latter, the uniqueness of the equilibrium outcome based on (perfect) equilibrium refinement does not extend to menu auctions (generally, equilibrium selection concepts are required). The descending proposals game resembles a Dutch auction, and thus, the outcome equivalence that we establish relates to the revenue equivalence of first–price, sealed–bid auctions and Dutch auctions.

### 2.1 Options and Preferences

Player 0 is the formateur and tries to form a winning coalition with a subset of the players $N = \{1,\dots,n\}$. The set of possible *programmes* (political platforms) is denoted $S$. The set of coalitions that can implement at least one programme $s \in S$ is denoted $C \subseteq \mathcal{P}(N)$, where $\mathcal{P}$ denotes the power set. The set of feasible programme–coalition combinations $(s,c)$ is denoted $Q_0 \subseteq S \times C$; generally, it would not be equal to $S \times C$, as not all coalitions need be able to implement all programmes. Moreover, our definition $C$ applies equally to unicameral legislatures, bicameral legislatures, and presidential democracies, i.e. we analyze these differing institutional settings in a unified framework. We refer to each element $q \in Q_0$ as an *option*; $s(q)$ denotes the programme and $c(q)$ denotes the set of players whose agreement player 0 requires to implement option $q$. The valuation functions $v_i : Q_0 \to \mathbb{R}$ map the set of options to the reals (for all players, including the formateur).

Player 0 has at least one outside option (the status quo), i.e. an option $q \in Q_0 : c(q) = \emptyset$ where 0 does not require the support of any $i \in N$. In equilibrium, player 0 would only choose options $r'$ for which there is no preferred outside option. Thus, only an option in the following set would be chosen along an equilibrium path.

$$\mathcal{Q} = \{q \in \mathcal{Q} \mid \forall q' : \quad v_0(q') > v_0(q) \ \Rightarrow \ c(q') \neq \emptyset\}. \tag{1}$$

To simplify the notation, we restrict our attention to $\mathcal{Q}$. By using the above notation, we implicitly assume (following Romer and Rosenthal, 1978) that the valuation of the outside option would be independent of the strategies played under formateur 0. In this way, we can not model that there is a possibly infinite sequence (protocol) of formateurs, which would end only if a formateur is able to implement an option other than the outside option. If the protocol is finite (e.g. if the term of legislature is finite), then the equilibrium play can be backward induced to satisfy our assumption (as illustrated, e.g., in Austen-Smith and Banks, 1988). Infinite models are considered explicitly below, in the “Applications” section, and there we show how to generalize the above simplification.

The valuation of some player $i$ may depend on the programme $s$ as well as on the coalition implementing $s$. This may apply equally to parties inside the coalition as well as to those outside the coalition, since their roles in the opposition (or, in the party hierarchy, or in any other “after–market”) would be different. We assume that, for any pair of options, if one player is not indifferent between the options, then no player is indifferent. Formally,

$$\forall q, q' \in \mathcal{Q} : \qquad \exists i : v_i(q) \neq v_i(q') \quad \Rightarrow \quad \forall i : v_i(q) \neq v_i(q'). \tag{2}$$

We refer to this assumption as *generic valuations*. Our main motivation of it is the after–market of a parliamentary decision, which is payoff–relevant as the set of players is rather small. For instance, if the players are parties and one party is affected by a decision, then the other parties would be affected indirectly as well. This assumption is somewhat relevant for our uniqueness result, but note that related assumptions are generally made in the literature (for instance, in cases of indifference between accepting and rejecting a given proposal it is assumed that the players would accept it, see Eraslan, 2002).

In the following, we shall not distinguish options that all players find equivalent. We therefore define a derived set of options $\mathcal{R}$, where each of the derived options $r \in \mathcal{R}$ actually comprises a class of payoff equivalent options $q \in \mathcal{Q}$. Let $Q \subseteq \mathcal{Q}$ denote a class of payoff equivalent options, and let $r \in \mathcal{R}$ denote the corresponding derived option. In the following, $C(r)$ denotes the set of coalitions that are required to implement at least one of the options in $Q$, and $S(r)$ denotes the set of programmes that are supported in $Q$. Note that we do not require $Q = S(r) \times C(r)$. Finally, let $c(r) = \cup_{c \in C(r)}\, c$ denote the set of players that are required for at least one option in $Q$.

The domain of the valuation functions $v_i$ is extended to $\mathcal{R}$. We will refer to the elements of $\mathcal{R}$ simply as options, though they are (by construction) sets of equivalent options. Thanks to the construction of $\mathcal{R}$, we know for all players $i$ (including the formateur) and all option pairs $r \neq r'$ that $v_i(r) \neq v_i(r')$. Based on the valuations of player 0, we finally define an ordering $\geq$ over the set of (derived) options $\mathcal{R}$.

$$\forall r_1, r_2 \in \mathcal{R}: \qquad r_1 \geq r_2 \quad \Leftrightarrow \quad v_0(r_1) \geq v_0(r_2) \tag{3}$$

Apparently, the ordering $\geq$ is complete, reflexive, and transitive. We also define $r_1 = r_2$ as being equivalent to $r_1 \geq r_2$ and $r_2 \geq r_1$, and $r_1 > r_2$ as being equivalent to $r_1 \geq r_2$ and $r_1 \neq r_2$.

## 2.2 The sealed offers game

The formateur asks the other players which options $r \in \mathcal{R}$ they would support. That is, for each class of payoff equivalent options, the formateur asks the players whether they would support an option in this class. He would have to choose the option (out of this class) that he actually implements only later, since the supporting players are indifferent with respect to the result of this choice. Each non–formateur player $i \in N$ offers to implement a subset of $\mathcal{R}_i = \{r \in \mathcal{R} \mid i \in c(r)\}$, i.e. of the options where $i \in c(r)$. Subsets of $\mathcal{R}_i$ are denoted $R_i$, and the strategy set of $i$ is $\mathcal{P}(\mathcal{R}_i)$. All players are completely informed.

Fix a strategy profile $R = (R_i)$. The indicator $A(r|R) \in \{0,1\}$ describes whether option $r$ can be implemented by the formateur, i.e. $A(r|R) = 1$ iff $\exists c \in C(r)\ \forall i \in c : r \in R_i$. Thus, player 0 will implement $r^*(R) = \max\{r \in \mathcal{R} \mid A(r|R) = 1\}$ in any perfect equilibrium. As this choice is unique for each offer profile $(R_i)$, we can take it as given. Thus, only the moves of the non–formateur players $i \in N$ are strategically relevant, and we restrict our attention to a game of the players $i \in N$. The payoff of player $i$ is $\pi_i(R) = v_i\big(r^*(R)\big)$. The strategy profile $R = (R_1, \ldots, R_n)$ is a Nash equilibrium if no player would be better off deviating unilaterally, i.e. if

$$\forall i\ \forall R_i' \subseteq \mathcal{R}_i: \qquad \pi_i(R) \geq \pi_i\left(R_i', R_{-i}\right). \tag{4}$$

A mixed strategy of player $i$ maps $\mathcal{R}_i$ to $[0,1]$. It is a collection of independent probability measures, each indicating the probability of the event that $i$ will offer his participation under a given $r \in \mathcal{R}_i$. Note that we understand the acts of accepting different options as independent tasks. We denote the probability that $i$ offers $r$ as $m_i(r)$. The probability that $i$ plays the pure strategy $R_i \subseteq \mathcal{R}_i$ under $m_i$ is

$$\mu(R_i|m_i) = \prod_{r \in R_i} m_i(r) * \prod_{r \in \mathcal{R}_i \setminus R_i} \big(1 - m_i(r)\big). \tag{5}$$

The expected payoff of $i$ under the mixed strategy profile $m = (m_i)$ is

$$\pi_i(m) = \sum_{(R_1,\ldots,R_n) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_n} v_i(R_1, \ldots, R_n) * \prod_{i \in N} \mu(R_i|m_i). \tag{6}$$

The definition of Nash equilibria in mixed strategy applies as above. Let $\boldsymbol{\varepsilon}$ denote a profile $(\varepsilon_{i,r})$ defining a real number for all $i \in N, r \in \mathcal{R}_i$. We say that $(m_i)$ is an $\boldsymbol{\varepsilon}$–equilibrium when it is a Nash equilibrium in the strategy space restricted to mixed strategies satisfying $m_i(r) \geq \varepsilon_{i,r}\ \forall i \in N \forall r \in \mathcal{R}_i$.

The profile $(m_i)$ is called trembling–hand perfect equilibrium (TPE) if it is a limit of $\boldsymbol{\varepsilon}$–equilibria for some sequence of profiles $(\boldsymbol{\varepsilon})$ approaching $\mathbf{0}$.

We will iteratively eliminate strategies until a unique strategy profile remains. The strategies that we eliminate are (strictly) dominated under full support, and thus, the unique undominated strategy profile is the unique perfect equilibrium. The idea underlying our arguments is that we can induce whether a player accepts option $r$ without requiring knowledge of which options $r' > r$ would be accepted by any of the players (under full support). Formally, for any $r$, any $i \in c(r)$, and any pair of completely mixed strategies $m_i^1, m_i^2$ that differ only in the probability assigned to accepting $r$, we can show that $m_i^1$ is better than $m_i^2$ in response to some $m_{-i}$ iff a condition is satisfied that does not depend on any of the probabilities assigned to options $r' > r$. As a result, when the arguments of iterative dominance have lead to unique acceptance probabilities for all options $r' < r$, we know everything that is required to induce whether $i$ would accept $r$.

Thus, the induction proceeds as follows. The outside option is denoted $r_0$. We can first induce which players would accept the option $r_1 = \min\{r' | r' > r_0\}$, as this would depend only on whether the outside option would be accepted (which is trivially true). Based on this, we can induce which players would accept $r_2 = \min\{r' | r' > r_1\}$, then which players would accept $r_3 = \min\{r' | r' > r_2\}$, and so on. Technically, we would require only a concept of rationalizability under full support to derive the solution, e.g. cautious rationalizability (Pearce, 1984), but under perfectness, some notation is simplified.

Fix $r \in \mathcal{R}$. Assume that the equilibrium strategies have already been induced for all $r' < r$ and assume that the induced equilibrium probabilities would be degenerate. Let $A_i(r') \in \{0,1\}$ denote the probability that $i$ offers $r'$, for all $r' < r$ where $i \in c(r')$. If restricted to options $r' < r$, then player 0 would choose $g(r)$ as defined next.

$$g(r) = \max\{r' < r \mid \exists c \in C(r') \, \forall i \in c : A_i(r') = 1\} \tag{7}$$

We will show that it is dominated to offer $r$ for a player $i$ who prefers $g(r)$ over $r$, and that it is dominant to offer $r$ when it is preferred to $g(r)$. Thus, the probabilities that $r$ is offered would be

$$\forall i \in c(r): \qquad A_i(r) = \begin{cases} 1 & \text{, if } v_i(r) > v_i(g(r)) \\ 0 & \text{, else.} \end{cases} \tag{8}$$

Note that Eq. (7), (8) provide a recursive definition of $A_i(r)$ for all $r$ and all $i \in c(r)$. Let $\mathcal{R}^*$ denote the set of TPEs.

**Lemma 2.1.** *Fix $r \in \mathcal{R}$. Assume that for all $r' < r$, the following holds.*

$$\forall R^* \in \mathcal{R}^* \; \forall i \in c(r'): \qquad A_i(r') = 1 \quad \Leftrightarrow \quad r' \in R_i^* \tag{9}$$

*Then, the equilibrium probabilities of accepting $r$ are degenerate for all $i \in c(r)$ and Eq.* (9) *holds for $r' = r$.*

**Example 1**

| $(s,c)$ | $v_0$ | $v_1$ | $v_2$ |
|---|---|---|---|
| $(x,\{1\})$ | 3 | 2 | 3 |
| $(y,\{2\})$ | 2 | 1 | 1 |
| $(z,\emptyset)$ | 1 | 3 | 2 |

**Example 2**

| $(s,c)$ | $v_0$ | $v_1$ | $v_2$ |
|---|---|---|---|
| $(x,\{1\})$ | 3 | 2 | 1 |
| $(y,\{2\})$ | 2 | 1 | 3 |
| $(z,\emptyset)$ | 1 | 3 | 2 |

The proof of Lemma 2.1 and selected further proofs are relegated to the appendix. By an inductive argument, we can now show that the above defined $A_i(r)$ completely characterize the unique perfect equilibrium. As a result, the set of options for which all required players offer their participation in equilibrium is

$$F = \{r \in \Omega \mid \exists c \in C(r') \, \forall i \in c : A_i(r) = 1\} \tag{10}$$

Player 0 will choose the one that he prefers most, $\max F$.

**Theorem 1.** *The trembling–hand perfect equilibrium of a generic sealed offers game is unique. It implies that* $\max F$ *results, based on* $F$ *defined through Eq.* (7), (8), *and* (10).

*Proof.* Eq. (9) holds for $r' = r_0 = \min \mathcal{R}$, since $c(r_0) = \emptyset$. Moreover, we know that if it holds for all $r' < r$, then it also holds for $r' = r$ (Lemma 2.1). Hence, we can induce that (9) holds for all $r' \in \mathcal{R}$. As a result, the set $F \subseteq \mathcal{R}$, as defined in (10), contains the (unique) set of options that player 0 can implement in equilibrium, and $\max F$ is chosen. □

*Example* 1. The (perfect) equilibrium of the sealed offers game implies $(z,\emptyset)$. It is weakly dominated for player 2 to offer participation under $(y,\{2\})$, and strictly so under full support. Hence, player 1 would not offer his participation under $(x,\{1\})$. In this example, we can find a second Nash equilibrium. In this equilibrium, both players offer their respective options. Player 2 can offer $(y,\{2\})$ as he would be outbid, and the strategy of 1, to offer $(x,\{1\})$, is his best response. This equilibrium is weak, as player 2 can deviate without costs, and under full support, he would deviate.

*Example* 2. Option $(x,\{1\})$ results in the sealed offers game. It is dominant for player 2 to offer participation under $(y,\{2\})$, implying that 1 would offer $(x,\{1\})$.

## 2.3 Equivalence to the Descending Proposals Scenario

We now show that the sealed offers game is outcome equivalent to a "Dutch auction" as defined in the following. The formateur proposes options from $\mathcal{R}$ to the players until one is accepted by a sufficient number of players, or until he runs out of options. Since the elements $r \in \mathcal{R}$ are actually "classes" of equivalent options, the formateur would propose an appropriate instance of this class (i.e. if there is an instance that would be accepted, then he would choose it). The options $r$ are (to be) proposed in descending order, i.e. the formateur does not act strategically here; he starts with the the option that he prefers most, he would end with the one that he prefers least, and he does not skip options. The most recent proposal of the formateur defines the state of the game. The set of states is $\Omega = \mathcal{R}$, it inheres all characteristics of $\mathcal{R}$. In each state $\omega \in \Omega$, the players in $c(\omega)$ vote "yes" (1) or "no" (0). If all relevant players vote yes, the game ends, and each player $i$ realizes the payoff $v_i(\omega)$.

The set of states where player $i$ has to vote is $\Omega_i = \{\omega | i \in c(\omega)\}$ and $i$'s strategy is a function $\sigma_i : \Omega_i \to \{0,1\}$. By this definition, the strategies are Markov; they depend only on the payoff–relevant details of the history of play (and not on who was rejecting which proposal). The assumption of Markov strategies is not restrictive and will be discussed below. The game starts in state $\max\Omega$. Under the strategy profile $\sigma = (\sigma_i)$, and given the current state is $\omega$, player $i$ realizes the payoff (defined recursively)

$$\pi'_i(\omega|\sigma) = \begin{cases} v_i(\omega) & \text{, if } \exists c \in C(\omega)\ \forall j \in c : \sigma_j(\omega) = 1 \\ \pi'_i(\max\{\omega' | \omega' < \omega\}|\sigma) & \text{, otherwise.} \end{cases} \tag{11}$$

Note that this payoff function is denoted $\pi'$, while the payoff function of the sealed offers game was denoted as $\pi$. Note also that in $\omega_0 = \min\Omega$, which is the outside–option of the formateur, no player can vote, and thus $\sigma_j(\omega_0) = 1 \forall j \in c(\omega_0)$ is satisfied for all strategy profiles. The strategy profile $\sigma$ induces a Nash equilibrium in state $\omega$ if no player is better off deviating unilaterally, i.e. if

$$\forall i\ \forall \sigma'_i : \qquad \pi_i(\omega|\sigma) \geq \pi_i(\omega|\sigma'_i, \sigma_{-i}), \tag{12}$$

and it induces a Markov perfect equilibrium (MPE) if it induces Nash equilibria in all states $\omega$. Mixed strategies are functions $m_i : \Omega_i \to [0,1]$ that describe for each state the probability that $i$ votes yes. The sets of mixed strategies are equivalent to those of the sealed offers game, and apart from a conditional probability calculation, this applies to the payoff function as well. In state $\omega$, the formateur would propose the option (out of class $\omega$) that is most likely to be accepted; let $\Pr(\omega|m)$ denote the probability that it is accepted. The expected payoff under strategy profile $m$ in state $\omega$ is

$$\pi'_i(\omega|m) = \sum_{\omega' \leq \omega} v_i(\omega') * \Pr(\omega'|m) * \prod_{\omega'' : \omega \geq \omega'' > \omega'} \left(1 - \Pr(\omega''|m)\right). \tag{13}$$

Thus, we assume patient players; the results hold equivalently under discounting for discount factors near 1. A strategy profile is a TPE if it induces a perfect Nash equilibrium in state $\max\Omega$ (as perfectness is defined above). Implicitly, a TPE induces perfect Nash equilibria in all states $\omega \in \Omega$, and thus, any TPE is an MPE (see also Selten, 1975).

**Proposition 2.2.** *There is a unique TPE in any generic descending proposals game, and the equilibrium is outcome equivalent to that of the sealed offers game. Formally, let $R^*$ denote the TPE of the sealed offers game and $\sigma^*$ any TPE of the descending proposals game, then*

$$\forall r \in \mathcal{R}\ \forall i \in c(r) : \qquad \sigma^*_i(r) = 1 \quad \Leftrightarrow \quad r \in R^*_i. \tag{14}$$

*Proof.* Fix a mixed strategy profile $m$ of the descending proposals game. It can also be understood as a mixed strategy profile of the sealed offers game. In the initial state $\max\Omega$, the expected payoff under $m$ is equivalent to the expected payoff of the sealed offers game under $m$ (as the payoff functions are equivalent then). Hence, for each perfect equilibrium of the sealed offers game, there must be a corresponding strategy profile that induces a perfect (Nash) equilibrium in state $\max\Omega$, and vice versa. Since the perfect equilibrium of the sealed offers game is unique, there also is a unique perfect equilibrium in state $\max\Omega$, and thus, a unique TPE. $\square$

Above, we concentrated on Markov strategies, and as a result of that, the (mixed) strategy space of the descending proposals game is equivalent to that of the sealed offers game. The restriction to Markov strategies is not necessary for the derived outcome equivalence, however. We show next that the unrestricted perfect equilibrium (UPE) is unique, too, i.e. there is a unique perfect equilibrium in the game where the strategies may depend on the complete history of play (i.e. on who rejected which proposals).

Consider an arbitrary state $\omega$ and let $\mathcal{R}_\omega = \{\omega' | \omega' \leq \omega\}$ denote the set of options that still can result in state $\omega$. As shown above, for all $\omega$ there are unique TPEs in the (descending proposals or sealed offers) game restricted to $\mathcal{R}_\omega$. Now consider the final state $\omega_0 = \min\Omega$. There is a unique strategy profile that induces a perfect Nash equilibrium in the game restricted to $\mathcal{R}_{\omega_0}$, and thus, all UPEs must induce it in all subgames implying state $\omega_0$—regardless of the history of play. Next, consider the state $\omega_1 = \min\{\omega | \omega > \omega_0\}$. Since the UPE payoffs in state $\omega_0$ are unique, they can not depend on the history of play in state $\omega_1$. Hence, the restriction to Markov strategies is strategically irrelevant in the descending proposals game starting in state $\omega_1$, and the uniqueness of the TPEs extends to UPEs in the descending proposals game restricted to $\mathcal{R}_{\omega_1}$. In this way, we can next induce the uniqueness for state $\omega_2 = \min\{\omega | \omega > \omega_1\}$, and iteratively for all other states.

## 2.4 Equivalence to the Constructive Proposals Scenario

The outcome equivalence extends to games where the options can be proposed in a rather loose order. The order is restricted only by a requirement of constructiveness (to avoid “blame game” politics, see Groseclose and McCarty, 2001). This rules out a number of subgame–perfect equilibria where the formateur threatens to repeat a single proposal (or, some set of proposals) until it is (or, one is) accepted. Basically, we assume that the reputation of unconstructive formateurs would suffer prohibitively (sooner or later), as a result of which threats of perpetual unconstructive behavior become incredible. We say that a proposal is *constructive* if it has never been proposed before or if it constitutes a compromise with respect to the previous proposal (i.e. if it is less preferred by the formateur). We assume that a formateur can not make a single unconstructive proposal (without being blamed significantly), but this assumption could be relaxed to allowing for any finite number of unconstructive proposals. Also, the formateur needs not be always able to come up with new proposals; the only important feature is that he is always able to make compromise proposals (of his choice) without being blamed of being unconstructive. We say that the assumption induces strategies in *constructive proposals*.

In a technical sense, the assumption of constructiveness relates to the assumption of *stationary* strategies in majority bargaining games (e.g. Baron and Ferejohn, 1989, and Eraslan, 2002). Both assumptions imply uniqueness of the equilibrium outcomes, only the motivations differ (a reference to the bargaining audience in our case, and simplicity in the case of stationarity). To underline this, let us otherwise align the following model to those assumed in the literature. On one hand, we assume (here) that the players respond *sequentially* to the proposal of the formateur (note, though, that simultaneous responses plus trembling–hand perfectness would lead to the same predictions). The formateur makes a new proposal after the first rejection of the standing proposal by any of the required players. On the other hand, we explicitly assume impatience in the sense of discount factors $\delta_i$ less than but close to 1 (we skip a formalization of the lower bound; our results also hold for patient players). Under more significant discounting, the equilibrium outcome would still be unique, but the outcome equivalence to sealed offers games can not be guaranteed (typically, the formateur would be better off).

A history of proposals is denoted $h^t = \left(h^t_1, h^t_2, \ldots, h^t_t\right) \in \mathcal{R}^t$. For $t' \leq t$, the term $h^t_{1\ldots t'}$ denotes the respective sub–history of proposals. We use $H^t$ to denote the set of $t$–round histories that are feasible under constructiveness. By definition, $H^1 = \mathcal{R}$. For all $t \geq 2$, $h^t \in H^t$ iff $h^t_{1\ldots t-1} \in H^{t-1}$ and

$$h^t_t < h^t_{t-1} \qquad \text{or} \qquad \forall t' < t:\ h^t_{t'} \neq h^t_t. \tag{15}$$

The set of all feasible histories is $H = \cup_t H^t$. The formateur's strategy describes a feasible proposal for the initial node $(\emptyset)$ and for any history that does not end with the outside option. Let us denote these histories as $H_0$, and the respective $t$–round histories as $H^t_0$. The strategy is a function $\sigma_0 : H_0 \to \mathcal{R}$ subject to $\forall h \in H_0 : (h, \sigma_0(h)) \in H$. Similarly, let $H_i$ denote the histories that end with an option requiring the support of player $i \in N$. The strategy of $i \in N$ is a function $\sigma_i : H_i \to \{0,1\}$, describing whether $i$ accepts that finally proposed option. Here, we allow that the decision depends on the history of proposals, but without restricting the generality (as above), we assume that the decision is independent of who was rejecting which of the previous proposals. For a given strategy profile $\sigma$, let $a(h^t|\sigma) \in \{0,1\}$ indicate whether all required players support option $h^t_t$. Formally, $a(h^t|\sigma) = 1$ iff $\exists c \in C(h^t_t)\ \forall i \in c : \sigma_i(h^t) = 1$. Recursively defined, the payoff of $i$ under $\sigma$ after history $h^t \in H^t_0$ is $(\forall t)$

$$\pi''_i(h^t|\sigma) = \begin{cases} v_i(h^t_t) & \text{, if } a(\sigma_0(h^t)|\sigma) = 1, \\ \delta_i * \pi''_i\big((h^t, \sigma_0(h^t))|\sigma\big) & \text{, otherwise.} \end{cases} \tag{16}$$

The definitions of Nash and subgame–perfect equilibria (SPEs) apply as usually.

**Theorem 2.** *Along the path of play, any SPE $\sigma^*$ of a generic constructive proposals game induces the unique TPE outcome of the corresponding descending proposals game.*

# 3 Applications

We now analyze variants of coalitional bargaining models proposed in the literature. The models that we cover are (originally) models of several formateurs; i.e. if the proposal of one formateur is rejected, then a new formateur is recognized randomly. We vary these models only with respect to the assumed communication of the formateur with the other players. Formally, we assume that each formateur conducts the *sealed offers game*, but as shown above, the results extend to the constructive proposals scenario. According to the latter, the formateur would be replaced only if he steps down (by choosing the outside option) or if he becomes unconstructive. We assumed throughout that the value of the outside option would not be subject to strategic considerations. The following shows that this assumption is comparably unrestrictive in our cases.

The following analysis relies on a corollary of Theorem 1: an option can not result in equilibrium if there is another option that the formateur and all required voters prefer (anything else can easily be led to a contradiction). We concentrate on this (negative) part of the equilibrium induction (ruling out options that can not result); we skip the explicit construction of an equilibrium.

## 3.1 Division of Cakes with Majority Decisions

Baron and Ferejohn (1989), Eraslan (2002), and others model the division of cakes (size 1) through majority decisions.

**Definition 3.1 (Baron–Ferejohn Model).** The bargaining proceeds in rounds, until a majority agrees to a proposal of how to allocate the cake. In each round, a player is stochastically recognized to propose an allocation, and in response to this proposal, the players vote *yes* or *no*. The set of players is $N$ such that $|N| > 2$, the agreement of $q < |N|$ players is required to allocate the cake (including the formateur), and the probability that $i$ is recognized as formateur is $p_i \in (0,1)$. The recognition probabilities are independent of the round $t$.

In the original model, the equilibrium outcome depends on the recognition probabilities (which does not extend to our variant of it). In the case of uniform recognition probabilities, the player recognized as formateur gets $1 - \frac{\delta(q-1)}{n}$, where $\delta$ is a discount factor, $q-1$ further players get $\frac{\delta}{n}$, and the remaining players get zero. If $n$ is odd and $q = \frac{1}{2}(n-1)$, then the formateur's payoff is in $\left(\frac{1}{2}, \frac{2}{3}\right)$.

In our variant of this model, there is a smallest monetary unit $\varepsilon > 0$ and all cake shares have to be integral multiples of $\varepsilon$. Thus, the option set is finite. We are interested in the equilibrium allocations when $\varepsilon$ is close to 0; the analysis applies, however, whenever $\varepsilon < \frac{p_i}{nq}\ \forall i$. In this model, the players are indifferent with respect to the coalition that supports a given proposal. Thus, the various classes of payoff–equivalent options differ only with respect to the induced allocation, any majority coalition can implement any option. We skip a formalization of the sets of feasible majority coalitions in the option set, and the accordingly simplified set of options is

$$\mathcal{R} = \left\{ x \in \mathbb{N}_0^N \mid \varepsilon * \sum_{j \in N} x_j \leq 1 \right\}. \tag{17}$$

Let $C$ denote the set of majority coalitions. Under option $r \in \mathcal{R}$, player $j$ is allocated $x_j(r)$ monetary units. As above, we assume that all players have generic valuation functions $v_i$, i.e. no player is indifferent with respect to any pair of options. We do not have to specify how the preferences are refined, though, as our results apply to any refinement that preserves the original order. Thus, they apply to all valuation functions satisfying

$$\forall r_1, r_2 \in \mathcal{R}, \forall j : \qquad x_j(r_1) > x_j(r_2) \quad \Rightarrow \quad v_j(r_1) > v_j(r_2). \tag{18}$$

Let $\rho$ denote the current formateur. Fix any equilibrium. First, we show that no option can result where a player other than the formateur gets $q\varepsilon$ or more. This is equivalent to showing, for any other $r \in \mathcal{R}$, that there is an alternative option $r'$ that a majority (including the formateur) prefers. That is, for all $r$,

$$\exists i \neq \rho : \; v_i(r) \geq q\varepsilon \qquad \Rightarrow \qquad \exists r' > r \; \exists c \in C \; \forall j \in c : \; v_j(r') > v_j(r) \tag{19}$$

Option $r'$ can be constructed in the following way. The coalition is made up of the $q-1$ players getting the smallest shares under $r$, plus the formateur. Thus, there is a player who gets at least $q\varepsilon$ under $r$ but is not in the new coalition. In $r'$, his share is zero, while his share is allocated such that all players of the new coalition get at least an $\varepsilon$ more than before (including the formateur). Consequently, all players of the coalition $c(r')$ prefer $r'$ to $r$, including the formateur, and thus, $r$ may not result in any (perfect) equilibrium of the sealed offers game.

Now, let $\pi_i$ denote $i$'s expected payoff ex ante (before the formateur is recognized), and let $\pi_{i|\rho}$ denote $i$'s expected payoff when $\rho$ is the formateur. The above implies that

$$\forall i \neq \rho : \qquad \pi_{i,\rho} < q\varepsilon. \tag{20}$$

This limits the expected payoff ex ante of player $i$. If $i$ is the formateur, then his maximal payoff is 1; if he is not the formateur, then his maximal payoff is less than $q\varepsilon$. Thus, his expected payoff satisfies (under the assumed bounds of $p_i$)

$$\pi_i < p_i * 1 + (1-p_i) * q\varepsilon < 1 - 2nq\varepsilon + 2nq\varepsilon * q\varepsilon = 1 - 2nq\varepsilon(1-q\varepsilon) < 1 - nq\varepsilon. \tag{21}$$

That is, $\forall i : \pi_i < 1 - nq\varepsilon$. Now, let $\rho$ be appointed as the formateur. We know that $\pi_\rho < 1 - nq\varepsilon$, and therefore, $\exists i \neq \rho : \pi_i > q\varepsilon$. Consequently, the outside option can not result in equilibrium (see above). This holds for each $\rho$: he reaches agreement and gets more than $1 - nq\varepsilon$. For $\varepsilon$ approaching zero, this is the whole cake.

**Proposition 3.2.** *Fix an equilibrium of the Baron–Ferejohn model with sealed offers and assume $\varepsilon \approx 0$. The proposal of the first formateur is accepted and he gets (approximately) the whole cake. The expected payoffs before the formateur is chosen approximate the recognition probabilities.*

Thus, any formateur is a dictator in this case. McKelvey (1976, 1979) has shown in a rather general framework that any agenda setter can become a dictator. His findings are somewhat related to this result, even though his voters are not acting strategically (farsightedly). Namely, McKelvey has shown that (almost) any two points in the space of options can be connected by consecutive majority votes. There is such a path in our model, too; it starts at the outside option and it ends at the equilibrium outcome (in the above case, it approximately ends at the dictator outcome). This path is made up by all proposals that the formateur can implement following the other players' offers. In our model, however, the path characterizes the equilibrium of a game, and thus, only specific paths can result. Therefore, the existence of paths that are compatible with our model and lead to the dictator outcome is not implied by McKelvey's results. The path restrictions imposed through our model are that the formateur's valuations must be monotonically increasing along the path and that the path must imply the smallest possible steps that lead to improvements for the formateur. One can say, though, that our model *implements* a McKelvey path in the above game (note that it is implemented in reverse order, we start with the dictator outcome and it is accepted as a majority prefers it to the next proposal).

## 3.2 A Legislative Voting Game

Jackson and Moselle (2002) discuss a model where the players vote on two–dimensional proposals.

**Definition 3.3 (Jackson–Moselle Model).** In each round a formateur is chosen randomly (using constant recognition probabilities $p_i$), he makes a proposal, and all players vote on it. To implement an option, more than 50% of the players have to agree to it, including the formateur. The number of players is odd and satisfies $n \geq 3$. The proposal has a distributive dimension, describing how to allocate a cake of size 1, and it has an ideological dimension, describing a one–parametric definition of a political programme. The set of possible cake allocations is denoted $X$. The set of political programmes is $Y = [0, \overline{Y}]$.

For their most general results, Jackson and Moselle (2002) assume that the valuations $v_i$ would be separable in the following sense. For all pairs of options, $(x,y)$ and $(x',y')$, and for all $i \in N$,

$$v_i(x,y) > v_i(x,y') \qquad \Leftrightarrow \qquad v_i(x',y) > v_i(x',y'). \tag{22}$$

In addition, the valuations are single–peaked in $y$, i.e. for all $i,x$, the maximum of $v_i(x,y)$ in $y$ is attained at $y = \hat{y}_i$. Let $\hat{y}$ denote the median of $\hat{y}_i$ over all $i$. We can not directly compare the results of our model to those of Jackson and Moselle (2002), as their findings mainly concern existence of and randomization in equilibrium. Nonetheless, we obtain a rather illustrative result if we assume linear valuations (also defined in Jackson and Moselle, 2002).

$$v_i(r) = x_i(r) + \alpha * |y(r) - \hat{y}_i| \tag{23}$$

The value of $\alpha$ is positive, but not otherwise restrict. The restriction to symmetric scaling terms $\alpha$ is not necessary, but it simplifies the notation.

If $\overline{Y} = 0$, then the Baron–Ferejohn model results. We assume a smallest monetary unit $\varepsilon_x$ and smallest possible steps for political programmes $\varepsilon_y$. Thus, $X = \left\{x \in \mathbb{N}_0^n \mid \sum_{i \in N} x_i \varepsilon_x \leq 1\right\}$, and $Y = \left\{y \in [0, \overline{Y}] \mid \exists i \in \mathbb{N} : y = i * \varepsilon_y\right\}$. Let $C$ denote the set of majority coalitions. Again, the players do not care about the resulting coalition, which allows us to simplify the option set by leaving the possible coalitions out. As above, we assume perfectly refined preferences (genericity), but do not require a specific formalization; our results hold when the original preferences are not reversed.

Fix any equilibrium. Let $\bar{x}_i$ denote the mean cake share of player $i$ and let $\bar{y}$ denote the mean programme (both ex ante, before the formateur is recognized). Then, the expected payoff of $i$ satisfies the following inequality.

$$\pi_i \leq \bar{x}_i + \alpha * |\bar{y} - \hat{y}_i| \tag{24}$$

It is an equality only if all political programmes $y$ that might be accepted along the path of play are on the same side of $\hat{y}_i$, i.e. either all are on the left or all are on the right of player $i$. Based on this, we can argue similarly to above. On one hand, there can not be an equilibrium where the formateur (ex post) realizes a cake share $x_\rho$ that is less than $1 - nq\varepsilon_x$. For any such option, there is another option that all required players prefer; it can be constructed as above, keeping the programme $y$ constant. In particular, there generally is an option that all required players prefer to the outside option: it implies the mean programme and besides the formateur, it involves the $\frac{n-1}{2}$ players with the smallest expected cake shares. All other players realize a cake share of 0 in this option; their expected cake shares $x_i$ are allocated to the participating players. There is enough cake to redistribute (such that all participating players can be offered more than they realize under the outside option) when $\varepsilon_x$ is small enough, i.e. when $\varepsilon_x < \frac{1}{n}$.

On the other hand, we can limit the programme $y$ resulting under a given formateur. First, we show that it would be between the median programme $\hat{y}$ and the formateur's ideal programme $\hat{y}_i$. Without loss of generality, assume that the formateur is to the right of the median, $\hat{y} < \hat{y}_i$. Suppose there would be an equilibrium implying option $r$ such that the resulting programme is $y(r) > \hat{y}_i$. Then, we can construct an option $r' > r$ where all required players are better off. The supporting coalition is made up of the players $j$ with $\hat{y}_j < \hat{y}$, the cake allocation is the same as in $r$, and the political programme is $\hat{y}_i$. Hence, the initially assumed $r$ can not result in equilibrium. Similarly, there can not be an equilibrium implying an option $r$ with $y(r) < \hat{y}$. Here, the coalition supporting a deviation is made up of the players $j$ such that $\hat{y}_j > \hat{y}$, who are better off under programme $y = \hat{y}$.

Finally, we show that if the formateur $i$ is within the quartiles of the distribution of ideological positions, then he approximately attains a dictatorship. Again, let us assume $\hat{y} < \hat{y}_i$. There are $n_1$ players to his right, with $n_1 > \frac{n-1}{4}$. In order to implement an option, he requires $n_2 = \frac{n-1}{2} - n_1$ further votes, implying $n_1 > n_2$. Assume that there is an equilibrium where he would propose (and implement) an option $r$ that is to the left of him, $y(r) < \hat{y}_i$. This can be led to a contradiction if $\varepsilon_x$ is sufficiently small. For, there is an option $r'$, supported by a majority, that all required players prefer. The programme $y(r')$ is one step to the right of $y(r)$, i.e. $y(r') = y(r) + \varepsilon_y$. The cake share of $\rho$ is constant. The $n_1$ players to the right of $\rho$ each lose $k_1 = \left\lceil \frac{\varepsilon_y}{\alpha * \varepsilon_x} - 1 \right\rceil$ pieces of cake, and thus, they are still better off under $r'$. These cake pieces are allocated to $n_2$ other players (whose identities are irrelevant), and if $\varepsilon_x$ is small enough, then these other players support $r'$, too. This is possible if

$$\varepsilon_x \leq \frac{n_1 - n_2}{n_1 + n_2} * \left( \frac{1}{\alpha} \varepsilon_y \right). \tag{25}$$

Note that this is positive, as $n_1 > n_2$. If $\varepsilon_x$ satisfies this condition, then the required $n_2$ players can each be allocated $k_2 = \left\lfloor \frac{\varepsilon_y}{\alpha * \varepsilon_x} + 1 \right\rfloor$ pieces of cake, implying that they indeed prefer $r'$ even though the programme shifted to their disadvantage. Since there is a preferable option, the initially assumed $r$ can not be supported in equilibrium. As a result, only an $r$ is sustained where $y(r) = \hat{y}_\rho$. A similar argument shows that, if the formateur is outside the quartiles, then he can implement approximately the quartile that he prefers.

**Proposition 3.4.** *Consider an equilibrium of the Jackson–Moselle model with sealed offers, linear valuations, and a small monetary unit $\varepsilon_x \approx 0$. Any player recognized as the formateur reaches agreement (without delay) and realizes a cake share of approximately 1. If the formateur's position is within the quartiles of the distribution of ideological positions, then he will be able implement his most preferred political programme, otherwise he is able to implement the preferred quartile.*

# 4 Variations of the Model

## 4.1 Constructive Proposal Bargaining with Imperfect Commitments

The first model variation that we discuss unifies the above models: the formateur can revise rejected proposals (when he is constructive in the above sense), and before each round, he can ask the voters which options they would support. Their offers are not necessarily perfectly binding in this model, misleading communication is only costly: when player $i$ rejects offer $r$ after having announced that he would accept it, then he has to bear costs $k_{i,r} > 0$. It is not costly to accept a proposal without having signaled the agreement. The formateur can define the set of voters that he asks for offers himself. Following the offers of these players, he makes a proposal, and if it is accepted, then it is implemented. Otherwise, he may ask for new offers (the previous offers are canceled) and make a revised proposal. We will derive an outcome equivalence to the above games. We skip a formal definition of the game, as the argument is comparably straightforward (given the definitions and arguments from Section 2).

In particular, we can concentrate on proving the equivalence for the following simplified game: when proposal $r$ was rejected, then the formateur has to make a proposal $r' < r$ in the next round. We refer to this game as the generalized descending proposals game (in the following, abbreviated

Table 1: The space of models

| | misleading communicat. is prohibitively costly | misleading communicat. is costly | no communication |
|---|---|---|---|
| formateur can revise rejected proposals | equiv. to sealed–offers | equiv. to sealed–offers | equiv. to sealed–offers |
| *new* formateur is drawn after rejected proposal | equiv. to sealed–offers | (unmodeled) | Austen-Smith and Banks (1988) model family |
| game ends after rejected proposal | equiv. to sealed–offers | (unmodeled) | ultimatum majority bargaining |

to generalized DPG). The extension to the "generalized constructive proposals game" is equivalent to the proof of Theorem 2 (which is in the appendix), it only requires an induction in the space of games.

We prove an outcome equivalence of the generalized DPG to the descending proposals game (DPG). To do so, we also make an induction in the space of games. We start with the game restricted to the lowest option $r_0 = \min \mathcal{R}$, i.e. with the game that has the option set $\mathcal{R}^0 = \{r_0\}$. The next game will be the one restricted to options $\mathcal{R}^1 = \mathcal{R}^0 \cup \big\{ \min(\mathcal{R} \setminus \mathcal{R}^0)\big\}$, and generally $\mathcal{R}^{i+1} = \mathcal{R}^i \cup \big\{ \min(\mathcal{R} \setminus \mathcal{R}^i)\big\}$. The outcome equivalence is immediate for the game restricted to $\mathcal{R}^0$, as this game only has one option (the outside option).

We now consider the game restricted to $\mathcal{R}^{i+1}$, under the assumption that the equivalence holds for $\mathcal{R}^j \neq \mathcal{R}\ \forall j \leq i$. Let $r$ denote the outcome of the DPG restricted to $\mathcal{R}^{i+1}$, and let $r'$ denote the respective outcome in the subgame following a rejection of $r$. A majority prefers $r$ over $r'$. Under the induction assumption, if $r$ is offered in the generalized DPG, and if it were to be rejected, then $r'$ would result. A majority prefers $r$, and thus, regardless of the players' offers, a majority accepts it when it is proposed. In particular, this holds even if no majority had offered it; the remaining players did not commit to accept it, but they find themselves better off accepting it rather than rejecting it once it is proposed. As a result, in any equilibrium, $r$ results or an option that the formateur prefers. The latter can be led to a contradiction. Assume an option $r'' > r$ would result. In the DPG, no majority would accept it, implying that no majority prefers it. It would result in the generalized DPG if and only if a majority offers $r''$ and all players $i$ in the majority have $v_i(r'') > v_i(r) - k_{i,r''}$ (they are better off accepting it than rejecting it plus bearing the costs of misleading communication). Under full support, any such player $i$ with $v_i(r'') < v_i(r)$ is strictly better off not offering it. Such players exist in any majority, and hence, no appropriate majority would offer it in equilibrium.

This argument extends the outcome equivalence to a rather large space of models, as depicted in Table 1. In this table, we understand "ultimatum majority bargaining" as the corresponding extension of ultimatum bargaining (see Güth et al., 1982). Moreover, we understand "*new* formateur is drawn after rejected proposal" to indicate that the protocol of formateurs implies that no formateur can be chosen a second time when a proposal of him was rejected.

## 4.2 Parliamentary Decisions can be Revised

We now discuss a model where the formateur can revise accepted proposals (the formal treatment is relegated to the appendix). There are two ways of thinking about this case. On one hand, there is an agenda setter who can make proposals that would (if accepted) cancel previous decisions of the parliament. On the other hand, the revisions might take place behind the scenes. The formateur asks for offers, announces his current choice, and then he is made further offers from players who see that their payoffs would improve thus. The game ends when the formateur can not improve his current choice anymore. We refer to this model as the *ascending proposals game*; it is related to English auctions.

To illustrate its relevance, let us assume that decisions can be revised but the formateur threatens that he would not attempt such revisions. Apparently, the formateur can only benefit from a revision, and if a profitable revision is possible, then the initial threat of not attempting revisions would be incredible (imperfect). The same applies to the other players: they could threaten not to participate in revision rounds, but then, they would forego opportunities to improve on their expected payoffs. These threats would be incredible, too. Consequently, the ascending proposals game is the natural extension of the above model when revisions are merely *possible*. In reality, this extension may be weakened when decision making is costly, but if it is not prohibitively costly, then the main arguments would still apply (qualitatively).

Our model is stylized in that early decisions can be revised perfectly, they do not restrict later decisions. In a more general model, where only partial revisions are possible, similar results would be obtained, but some details of the argument would depend on the structure of possible revisions. We also assume that the players care only about the final decision (i.e. patience), but the results extend equivalently to discount factors close to 1. Finally, we assume that the players do not care about the end of the term of the formateur (i.e. the current formateur has an infinite time horizon). This is sufficient to show that the formateur may be worse off than above; if there would additionally be government changes, then this tendency would be even more significant (see, e.g. Baron, 1996, and Dixit et al., 2000).

In our formalization of this model, in each round a "sealed offers game" is played. The last successful proposal (or, the formateur's current favorite) defines the *state* of the game and serves as the status quo in the current round. Equivalently, we could assume that a constructive proposals game is played in each state (i.e. in each state, the formateur makes a sequence of proposals until one is accepted, then a new state is entered). The equilibrium play can be induced backwards in the space of possible states $\omega \in \Omega$. The induction starts in the state that is most preferable to player 0 $(\max \Omega)$ and ends with the least preferable one $(\min \Omega)$. For each state, we can induce the equilibrium actions as we did in the sealed offers game: starting with the state actions least preferable for player 0, ending with the actions most preferable for him. The equilibrium strategies are not generally unique (if the players are patient, then delay is possible), but the equilibrium outcome is generally unique.

The actual induction is in the appendix; here, we want to illustrate the relevance of the possibility of revisions. In particular, we want to answer the following two questions: Can an outcome equivalence be established between ascending offers and sealed offers games? And, is the ascending offers game always preferable for player 0? We do so based on the Examples 1 and 2.

*Example* 3 *(Example 1 revisited).* The (perfect) equilibrium of the sealed offers game implies $(z, \emptyset)$, as discussed above, while the equilibria of the ascending offers game imply $(x, \{1\})$. Thus, the predictions differ. In the ascending offers game, offering $(y, \{2\})$ leads to a state resulting in the endpoint $(x, \{1\})$. As player 2 prefers $(x, \{1\})$ to the outside option, it is weakly dominant for him to offer $(y, \{2\})$ (strictly under full support). In response, player 1 is best off offering participation under $(x, \{1\})$.

*Example* 4 *(Example 2 revisited).* This provides the opposite case, thereby showing that ascending offers games are not generally favorable for player 0. Option $(x, \{1\})$ results in the sealed offers game, see above, and $(z, \emptyset)$ results in the ascending offers game. Here, it is dominant for player 2 to offer participation under $(y, \{2\})$ in the sealed offers game, while it is dominated in the ascending offers game. For, in the ascending offers game, $(y, \{2\})$ leads to a subgame where $(x, \{1\})$ results and player 2 would be worse off than in the outside option. Thus, 2 is better off avoiding the state where $(y, \{2\})$ is the standing proposal.

In Example 2, the non–formateur players are better off in the outcome of the ascending offers game than in the outcome of the sealed offers game. They manage to reach this outcome by bidding strategically (farsightedly). Player 2 actually prefers $(y, \{2\})$ to the outside option, and if he would bid sincerely, he would offer it in the initial state. This deviation from sincere bidding is an instance of tacit collusion, and can be expected in a large number of such (multiple–round) voting games, depending on the preferences.

# 5 Discussion and Conclusive Remarks

We analyzed models of coalitional bargaining that were constructed to represent the processes in democratic voting institutions. Our framework allows that coalitions constitute majorities not simply if they control given voting shares. Thus, we unify models of single parliaments, bicameral democracies, and presidential democracies. We also allow that the veto rules and the required voting shares may depend on the proposal at hand. There is no restriction of the structure of the proposals (as a separability into distributive and ideological dimensions), nor of the preference functions (apart from genericity). The games that we analyzed are defined on discrete option spaces, which appears to be the most convenient choice in this case. The equilibrium outcomes are unique and are characterized through a simple program. In two well–known applications (the Baron–Ferejohn model and the Jackson–Moselle model), our model implies dictator outcomes along (reversed) McKelvey majority paths, but in general, the formateur can be far away from attaining dictatorship.

Given the discreteness of our option spaces, the outcome uniqueness may appear particularly surprising. In most other bargaining models (e.g. Rubinstein, 1982), discreteness of the option spaces leads to a plethora of equilibria. Our model differs from typical bargaining models, however, in that the formateur has to revise rejected proposals constructively. When we apply this to the Rubinstein model, assuming a constant formateur, then we can see that the corresponding outcome is unique, too. Interestingly enough, the resulting equilibrium does not favor the formateur at all. When the non–formateur player is sufficiently patient, then he would reject all proposals but the penultimate one (the last option where the formateur is still better off than in the outside option). For, he can induce that this proposal would be made (sooner or later) in any subgame. Hence, it is not immediate that a formateur as we modeled him is generally close to being a dictator. Nonetheless, it results in the examples of majority bargaining that we discussed above. In the following, we discuss the robustness of these results. We skip arguments that we touched already. This concerns, in particular, cases where parliamentary decisions can be revised (and extensions of this to possibly varying governments, as modeled in Dixit et al., 2000) and the rather obvious relevance of veto rights.

In our variant of the Rubinstein model, the formateur suffers from the lack of competition between the voters. In models of majority bargaining, i.e. when unanimous decisions are explicitly not required, competition arises rather naturally: the players can not demand higher cake shares, as the formateur would coalesce with other players then. This presumes, however, that a sufficient variety of coalitions exists. In the models of the literature, the player's valuations do not depend (significantly) on the resulting coalition; thus, the players are equally willing to participate in all coalitions. It is easy to think of circumstances, however, where the players care about their reputation and would therefore demand high compensations for participating in certain coalitions. In these case, the competition between the players may be obstructed severely, leading to more moderate results. Note that the generality of our framework allows for such effects.

Another objection to extreme results is related to observations in laboratory implementations of "ultimatum bargaining" games (for a survey, see Camerer, 2003). There, it is observed that players prefer outside options (zero payoffs for all) to payoff allocations extremely favoring the "proposer" (formateur). This may apply similarly to our model of coalitional bargaining, e.g. if certain ideological positions would never be given up for increased cake shares. Or, as it is modeled in ultimatum games, players do not accept extreme proposals because of equity concerns—all players should benefit comparably from a given proposal. Again, this can be modeled within our framework.

A similar argument applies if the formateur has no cake to allocate, e.g. if his proposals would be purely ideological (as in Primo, 2002, for instance). Apparently, competition as observed above can not arise then. For instance, one might argue, this applies to ministers. In reply, we would argue that ministers stay in office for rather lengthy periods of time, and in the corresponding supergame (with other cabinet members or coalition members), they might contingent their compromises in future debates on the support in the current one. In the behavioral literature, such contingencies are known as reciprocity (e.g. Rabin, 1993). Following this literature, we would say that the preferences for (or against) given coalitions can become significant in later stages of the game, depending on the coalitions' votes in earlier stages. As a result, the formateur (minister) has got something to allocate, implying that extreme results may be observed.

Finally, let us note that the formateurs and agenda setters are typically appointed by some principal (e.g. by prime ministers, presidents, chairmen, or even the people). The principal appoints a player as formateur because of this player's preferences, and only persons with specific preferences would be appointed as formateurs. If it can be induced that the formateur attains dictatorship following his appointment (as in the above models), then the players voting for his appointment can vote sincerely—they could simply vote for the candidate closest to their own preferences. More generally, the more powerful the formateur would be, the more sincerely the appointing players can vote for their favorites, and the less they have to vote strategically (as in Buchholz et al., 2005; for strategic delegation in general, see e.g. Fershtman and Judd, 1987). As a result, the formateur's power does not generally obstruct democratic processes, it may even simplify them. Models of complete election cycles (e.g. following Austen-Smith and Banks, 1988, and Diermeier et al., 2003) may shed more light on this issue.

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# A Decisions can be Revised (Formal Treatment)

The players' strategies depend on the proposal that was accepted most recently (the status quo). Payoff–irrelevant aspects of the history of play are neglected, without loss of generality (as in the descending proposals game). The set of states is denoted $\Omega = \mathcal{R}$, while $\omega \in \Omega$ denotes the most recently accepted proposal. The option set $\mathcal{R}$ has the same characteristics as the one used above. Fix a state $\omega \in \Omega$. For notational reasons, we assume that the players may not submit offers that concern options that player 0 does not prefer to $\omega$. Thus, the set of offers that $i$ can make in state $\omega$ is

$$\mathcal{R}_i(\omega) = \mathcal{R}_i \cap \{r | r > \omega\}. \tag{26}$$

The set of $i$'s strategies in state $\omega$ is $\mathcal{P}(\mathcal{R}_i(\omega))$. A Markov strategy of $i$ is a function $\sigma_i : \Omega \rightarrow \mathcal{P}(\mathcal{R}_i)$, subject to $\sigma_i(\omega) \subseteq \mathcal{R}_i(\omega)\ \forall\omega$. The set of $i$'s Markov strategies is denoted $\Sigma_i$. In response to a given action profile $\sigma(\omega) = (\sigma_i(\omega))$, player 0 implements the most preferable option and a transition to state

$$\tau(\sigma,\omega) = \max\left(\{\omega\} \cup \{r \in \mathcal{R} \mid \exists c \in C(r)\ \forall i \in c:\ r \in \sigma_i(\omega)\}\right) \tag{27}$$

occurs. If $\omega = \tau(\sigma,\omega)$, the game ends. We refer to all states satisfying $\omega = \tau(\sigma,\omega)$ as *endpoints* under $\sigma$. The endpoint that results (under $\sigma$) once state $\omega$ is reached is denoted $E(\omega|\sigma)$. Recursively defined, it is

$$E(\omega|\sigma) = \begin{cases} E\left(\tau(\sigma,\omega)|\sigma\right) & \text{, if } \tau(\sigma,\omega) \neq \omega, \\ \omega & \text{, otherwise.} \end{cases} \tag{28}$$

The payoff of $i$ under $\sigma = (\sigma_i)$, once state $\omega$ is reached, is $\pi_i(\sigma|\omega) = v_i\big(E(\omega|\sigma)\big)$. Thus, we assume patient players. The equilibrium outcome is robust to discounting when the discount factors $\delta_i$ are close to 1 (only possible delay is reduced). A profile $\sigma$ of Markov strategies induces a Nash equilibrium in state $\omega$ if $(\forall i)$ no unilateral deviation would be profitable.

$$\forall i \in N\, \forall \sigma_i' \in \Sigma_i : \qquad \pi_i(\sigma|\omega) \geq \pi_i(\sigma_i', \sigma_{-i}|\omega). \tag{29}$$

The profile $\sigma$ is a Markov perfect equilibrium if it induces Nash equilibria in all states $\omega \in \Omega$.

In state $\omega$, a mixed strategy of $i$ defines for each $r \in \mathcal{R}_i(\omega)$ a probability that $i$ will offer $r$. Notably, $\mathcal{R}_i(\omega)$ may be an empty set; then, player $i$ would only have the strategy of doing nothing (and can not randomize). The set of $i$'s mixed strategies in state $\omega$ is denoted $M_i(\omega)$. It is the set of functions $m_i(\omega) : \mathcal{R}_i(\omega) \to [0,1]$. The set of $i$'s mixed strategies for the whole game is $\bigtimes_\omega M_i(\omega) =: M_i \ni m_i$. A profile of mixed strategies is $m \in M$.

As defined above, let $\Pr(r|m,\omega)$ denote the probability that $r$ can be implemented under $m$ in state $\omega$ (i.e. that it would be offered by all required players). Then, the probability that state $\omega'$ immediately follows state $\omega$ is

$$\mu(\omega'|m,\omega) = \Pr(\omega'|m,\omega) * \prod_{r>\omega'} (1 - \Pr(r|m,\omega)) . \tag{30}$$

The probability that $\omega'$ will be the resulting endpoint, given the current state $\omega$, is (defined recursively)

$$PE(\omega'|m,\omega) = \begin{cases} 0 & \text{, if } \omega' < \omega \\ \sum_{\omega''>\omega} \mu(\omega''|m,\omega) * PE(\omega'|m,\omega'') & \text{, if } \omega' > \omega \\ \mu(\omega|m,\omega) & \text{, else.} \end{cases} \tag{31}$$

The expected payoff of $i$ under $m = (m_i)$ in state $\omega$ is

$$\pi_i(m|\omega) = \sum_{\omega' \in \Omega} PE(\omega'|m,\omega) * v_i(\omega'). \tag{32}$$

The definition of trembling–hand perfect Markov equilibria (TPEs) applies as above. In the following, $\Sigma^*$ denotes the set of pure TPEs. For the formal exposition, we restrict our attention to pure equilibria, but the outcome's uniqueness extends to mixed equilibria (while the equilibrium path leading to this outcome may be stochastic).

Fix a state $\omega \in \Omega$ and assume, for all $\omega' > \omega$, that the endpoint following a transition to state $\omega'$ would be unique (for all TPEs). Then, a function as the following exists.

$$\exists E' : \{\omega'|\omega' > \omega\} \to \Omega \quad \text{such that} \quad \forall \omega' > \omega, \forall \sigma^* \in \Sigma^* : \quad E'(\omega') = E(\omega'|\sigma^*) \tag{A1}$$

This function, describing the play in later states, allows us to induce the equilibrium strategies for state $\omega$. The equilibrium strategies will be characterized through "upper" and "lower" bounds (strategies within these bounds differ only with respect to whether there is delay). $A_{1,i}(r|\omega)$ denotes the minimal probability of the event that player $i$ offers option $r$ when the current state is $\omega$. It is based on a term $g_1(r|\omega)$ describing which option $r' < r$ would be implemented otherwise (conditional on the event that none of the options $r'' \geq r$ could be implemented). We show that player $i$ offers his participation under $r$ if (but not only if) he is better off in $E'(r)$ than in $E'(g_1(r|\omega))$.

$$\forall r > \omega, \forall i \in c(r) : \quad A_{1,i}(r|\omega) = \begin{cases} 1 & \text{, if } v_i(E'(r)) > v_i(E'(g_1(r|\omega))) \\ 0 & \text{, else.} \end{cases} \tag{33}$$

$$\forall r > \omega : \qquad g_1(r|\omega) = \max\left(\{\omega\} \cup \{r' < r \mid \exists c \in C(r')\ \forall i \in c : A_{1,i}(r'|\omega) = 1\}\right) \tag{34}$$

$A_{2,i}(r|\omega)$ denotes the respective maximal probability: player $i$ offers participation only if (but not if) he is not worse off in $r'$ than under the next–best option implementable for 0, i.e. under $E'(g_2(r|\omega))$. Note that we will show that $g_1(r|\omega)$ and $g_2(r|\omega)$ as constructed here are equivalent.

$$\forall r > \omega, \forall i \in c(r): \quad A_{2,i}(r|\omega) = \begin{cases} 1 & \text{, if } v_i(E'(r)) \geq v_i(E'(g_1(r|\omega))) \\ 0 & \text{, else.} \end{cases} \tag{35}$$

$$\forall r > \omega: \quad g_2(r|\omega) = \max\left(\{\omega\} \cup \{r' < r \mid \exists c \in C(r')\ \forall i \in c:\ A_{2,i}(r'|\omega) = 1\}\right) \tag{36}$$

This limits the sets of options that the formateur will be able to implement in state $\omega$. To formalize the following arguments, let $A_{\sigma,i}(r|\omega)$ denote the actual offer probabilities for a given strategy profile $\sigma \in \Sigma$.

$$\forall r > \omega: \quad A_{\sigma,i}(r|\omega) = \begin{cases} 1 & \text{, if } i \in c(r) \text{ and } r \in \sigma_i(\omega) \\ 0 & \text{, else} \end{cases} \tag{37}$$

Derived from the above, $F_k(r|\omega)$ denotes a set of options that are implementable and elements of $\{r'|\omega \leq r' \leq r\}$, for all $k \in \{1,2,\sigma\}$. For $k = 1$ it is the minimal set, for $k = \sigma$ it is the actual set, and for $k = 2$ it is the maximal one.

$$\forall r \geq \omega: \quad F_k(r|\omega) = \{\omega\} \cup \{r' \leq r \mid \exists c \in C(r')\ \forall i \in c:\ A_{k,i}(r'|\omega) = 1\} \tag{38}$$

The following lemmas comprise the main parts of the induction. Lemmas A.1 and A.2 state that all strategy profiles that are within the above bounds $(A_1, A_2)$ induce the same endpoint function (hence, the bounds are equivalent with respect to the induced endpoints, too). Lemma A.3 states that the equilibrium strategies must be within these bounds.

**Lemma A.1.** *Fix $\omega$ and assume* (A1) *applies. Next, fix $r \geq \omega$ and a set $F \subseteq \mathcal{R}$. Then,*

$$F_1(r|\omega) \subseteq F \subseteq F_2(r|\omega) \quad \Rightarrow \quad E'(\max F_1(r|\omega)) = E'(\max F) = E'(\max F_2(r|\omega)). \tag{39}$$

**Lemma A.2.** *Fix $\omega$ and assume* (A1) *applies. Fix $r > \omega$ and an arbitrary $A_{3,i}$ such that*

$$\forall r', \forall i \in c(r'): \quad \omega < r' \leq r \quad \Rightarrow \quad A_{1,i}(r'|\omega) \leq A_{3,i}(r'|\omega) \leq A_{2,i}(r'|\omega). \tag{40}$$

*Construct $F_k(r|\omega)$ for $k \in \{1,2,3\}$ according to Eq.* (38). *Then, $F_1(r|\omega) \subseteq F_3(r|\omega) \subseteq F_2(r|\omega)$.*

**Lemma A.3.** *Fix $\omega$ and assume* (A1) *applies. If $\sigma^* \in \Sigma^*$, then*

$$\forall r > \omega: \quad A_{1,i}(r|\omega) \leq A_{\sigma^*,i}(r|\omega) \leq A_{2,i}(r|\omega). \tag{41}$$

As a result, we can define a function $E^* : \Omega \to \Omega$ that equates with the (unique) endpoint function induced in all TPEs. Fix any $\omega$ and assume $E^*$ is defined for all $\omega' > \omega$. Iteratively (increasing $r$), this allows to define $f(r|\omega)$ for all $r \geq \omega$. First, define $f(\omega|\omega) = \omega$, and using $p_r = f(\max\{r'|r' < r\}|\omega)$,

$$\forall r > \omega: \quad f(r|\omega) = \begin{cases} E^*(r) & \text{, if } \exists c \in C(r)\ \forall i \in c:\ v_i(E^*(r)) > v_i(p_r), \\ p_r & \text{, else.} \end{cases} \tag{42}$$

This allows to define $E^*(\omega)$ as

$$E^*(\omega) = f(\max \Omega|\omega). \tag{43}$$

**Theorem 3.** *Fix a generic ascending offers game. For all $\omega \in \Omega$, each TPE $\sigma^*$ induces the (unique) endpoint $E(\omega|\sigma^*) = E^*(\omega)$ in state $\omega$, using $E^*$ as defined in Equations* (43), (42).

*Proof.* The claim holds in the state $\omega_1 = \max\Omega$. For, in the eyes of 0, $\max\Omega$ is the most profitable option, and in this state, the game ends as the players can not offer more preferred ones. Hence, the resulting option is unique in this state. Thus, (A1) holds for $\omega = \omega_2 = \max\{\omega'|\omega' < \omega_1\}$, based on which Lemmas A.1, A.2, A.3 show that the endpoint resulting in $\omega_2$ is unique (under perfectness), and that it is equivalent to $E^*(\omega_2)$. Thus, (A1) holds for $\omega = \omega_3 = \max\{\omega'|\omega' < \omega_2\}$, and iteratively, we can thus show that the above claim holds for all $\omega \in \Omega$. □

# B Relegated Proofs

*Proof of Lemma 2.1.* Fix an arbitrary completely mixed strategy profile $m$ satisfying

$$\forall r' < r \;\; \forall i \in c(r') : \qquad \begin{array}{ll} m_i(r') > 1-\varepsilon & \text{, if } A_i(r') = 1 \\ m_i(r') < \varepsilon & \text{, else} \end{array} \tag{44}$$

for some $\varepsilon > 0$. Fix $i \in c(r)$. First, we show that if $A_i(r) = 1$, then $A_{R^*,i} = 1$ holds for any TPE $R^*$ (by an argument of dominance). Fix an $i \in c(r)$ and consider the following two strategy profiles $m^1, m^2$.

- strategy $m^1_k(r') = m_k(r')$ for all $r', k \in c(r')$ except $m^1_i(r) = 1$
- strategy $m^2_k(r') = m_k(r')$ for all $r', k \in c(r')$ except $m^2_i(r) = 0$

Let $\Pr(r|m)$ denote the probability that $r$ can be implemented under $m$. Define $\pi^1 := \pi_i(m^1)$ and $\pi^2 := \pi_i(m^2)$. Then, there exists a $\pi \in \mathbb{R}$ such that

$$\pi^1 = \pi + \pi_A * \prod_{r'>r} \left(1 - \Pr\left(r'|m^1\right)\right) \tag{45}$$

$$\pi^2 = \pi + \pi_B * \prod_{r'>r} \left(1 - \Pr\left(r'|m^2\right)\right) \tag{46}$$

for appropriately defined conditional payoffs $\pi_A, \pi_B$. Thus, $\pi_1 > \pi_2$ is equivalent to $\pi_A > \pi_B$ for all $\varepsilon > 0$. These conditional payoffs are

$$\pi_A = \Pr\left(r|m^1\right) * v_i(r) + \left(1 - \Pr\left(r|m^1\right)\right)\pi_B \tag{47}$$

$$\pi_B = \sum_{r'<r} v_i(r') * \Pr\left(r'|m^2\right) * \prod_{r'':r>r''>r'} \left(1 - \Pr\left(r''|m^2\right)\right). \tag{48}$$

Let $\varepsilon'$ denote the conditional probability that a strategy profile leading to $g(r)$ is drawn. Then, there exists a $\pi'$ such that

$$\pi_B = (1-\varepsilon') * v_i(g(r)) + \varepsilon' * \pi'. \tag{49}$$

For each $\bar{\varepsilon}' > 0$ there exists an $\varepsilon > 0$ such that all mixed strategies constructed as above induce an $\varepsilon' < \bar{\varepsilon}'$. Hence, if $\varepsilon$ is sufficiently close to zero, then the implied $\varepsilon'$ is sufficiently close to zero such that $\pi_A > \pi_B$ is equivalent to $v_i(r) > v_i(g(r))$. This, in turn, is satisfied if $A_i(r) = 1$. Consequently, player $i$ is best off putting the maximal probability weight on offering $r$ in this case. To conclude, when $\varepsilon$ approaches 0, then all limits of completely mixed best responses (under the initial assumption, Eq. 9) imply that $i$ offers $r$ with probability 1 if $A_i(r) = 1$, i.e. $A_i(r) = 1$ implies $A_{R^*,i}(r) = 1$.

Similarly, we can argue in the second case to show that $A_i(r) = 0$ implies $A_{R^*,i}(r) = 0$, given Eq. (9) applies. In this case, $\pi_A < \pi_B$ is equivalent to $v_i(r) < v_i(g(r))$ for sufficiently small $\varepsilon$, which is satisfied if $A_i(r) = 0$. Finally, note that either maximal or minimal probability weight for offering $r$ is dominant, which implies that all equilibrium probabilities are degenerate. □

*Proof of Theorem 2.* The claim holds for the game restricted to the option set $\mathcal{R}_0 = \{\min \mathcal{R}\}$, as there is only the outside option in this game. We show inductively that the claim also holds for games with increasing option sets, leading to the game with option set $\mathcal{R}$. The sequence of option sets is denoted $\mathcal{R}_i$, for increasing $i$, and for each $i \geq 1$, it satisfies

$$\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{\min(\mathcal{R} \setminus \mathcal{R}_{i-1})\}. \tag{50}$$

Fix $i \geq 1$ and assume that the claim holds for all $j < i$. Thus, if 0's most preferred option in the game with option set $\mathcal{R}_i$ is rejected, then the resulting subgame is outcome equivalent to the descending proposals game with option set $\mathcal{R}_{i-1}$. Hence, the decision of whether to reject $\max \mathcal{R}_i$ is equivalent to the respective decision in the descending proposals game. If it is accepted in the descending proposals game, then the formateur offers it here, and it would be accepted, too. If it is not accepted in the descending proposals game, then two cases have to be distinguished. First, assume that there are differing equilibrium outcomes and the formateur prefers the descending proposals game equilibrium to the one implied under $\sigma^*$. Then, he can propose $\max \mathcal{R}_i$, which is rejected, in order to realize the descending proposals game outcome. If he is sufficiently patient, he is better off thus, implying that $\sigma^*$ is not an SPE. Secondly, if there are differing equilibria and the formateur is better off under $\sigma^*$ than in the descending proposals game outcome, then there is a player $i \in N$ who can deviate profitably (and unilaterally) from $\sigma^*$. Let $r$ denote the outcome of the descending proposals game, and $r'$ the outcome under $\sigma^*$. Since $r'$ does not result in the descending proposals game, there must be players $i \in c(r')$ that do not prefer $r'$ to $r$. If any of them would reject all offers of $r'$, then some $r''$ results. If $r''$ equals $r$, then the deviation to rejecting $r'$ was profitable. Similarly, he is better off in any subgame where an $r'' < r$ should result; then, formateur would deviate to proposing $\max \mathcal{R}_i$, to collect $r$. Finally, in a subgame where an $r'' > r$ results, there would be another player better off deviating such that $r$ results. In any case, and given he is sufficiently patient, the first player is better off deviating from accepting $r'$. Thus, the assumed $\sigma^*$ is not an equilibrium, implying that any equilibrium leads to the descending proposals game outcome. □

*Proof of Lemma A.1.* The claim obviously holds for $r = \omega$; in this case, $E'(\max\{\omega\}) = \omega$ results in all three cases. Now, fix an arbitrary $r > \omega$ and, to carry out the induction, assume that the claim holds for all $r' : \omega \leq r' < r$. It would not hold for $r$ only if $r \notin F_1(\omega)$ and $r \in F_2(\omega)$. By construction, this case implies

$$v_i(E'(r)) = v_i\left(E'\left(\max\{r' \in F_1(\omega) | r' < r\}\right)\right) \tag{51}$$

and by the induction's assumption, the right–hand side is equal to

$$= v_i\left(E'\left(\max\{r' \in F_2(\omega) | r' < r\}\right)\right). \tag{52}$$

When the payoffs are equal, then the endpoints must be equal, too, i.e.

$$E'(r) = E'\left(\max\{r' \in F_1(\omega) | r' < r\}\right) = E'\left(\max\{r' \in F_2(\omega) | r' < r\}\right). \tag{53}$$

Hence, the claimed relation holds for $r$, as well. Iteratively, we can thus show that the claim holds for all $r \geq \omega$. □

*Proof of Lemma A.2.* For all $k \in \{1,2,3\}$, we have $\omega \in F_k(r|\omega)$ by construction. We have to show that, for all $r'$ satisfying $\omega < r' \leq r$,

$$F_1(r|\omega) \cap \{r'\} \subseteq F_3(r|\omega) \cap \{r'\} \subseteq F_2(r|\omega) \cap \{r'\}. \tag{54}$$

Assume that this does not hold for some $r'$. There are two possible cases to be distinguished; first, we consider $F_1(r|\omega) \cap \{r'\} \not\subseteq F_3(r|\omega) \cap \{r'\}$. In this case, there exists an $i \in c(r')$ such that $A_{1,i}(r'|\omega) = 1$ and $A_{3,i}(r'|\omega) = 0$. This implies $A_{1,i}(r'|\omega) \not\leq A_{3,i}(r'|\omega)$, which contradicts the lemma's assumption. Similarly, $F_3(r|\omega) \cap \{r'\} \not\subseteq F_2(r|\omega) \cap \{r'\}$ contradicts the lemma's assumption, and by transitivity, (54) results. This holds for all $r'$, and thus, the lemma results. □

*Proof of Lemma A.3.* This can be proved similarly to Lemma 2.1, and is therefore abbreviated. The only differences are that we can construct mere bounds of the equilibrium strategies, and that in the definitions of $\pi_{1|A}$ and $\pi_{2|A}$, the terms $v_i(r)$ and $v_i(r')$ have to be substituted with $v_i(E'(r))$ and $v_i(E'(r'))$, respectively. As above, we have to proceed iteratively. We can show that the claim holds for any $r > \omega$ if it holds for all $r'$ satisfying $\omega < r' < r$, and it trivially holds in $r = \omega$. Define $r^o := E'(g_1(r|\omega))$.

On one hand, if $A_{1,i}(r|\omega) = 1$, then $A_{\sigma^*,i}(r|\omega) = 1$ is claimed to be implied. As above, under full support we can show that player $i$ would be best off putting maximal probability weight on offering $r$ if $v_i(E'(r)) > v_i(r^o)$. This holds if $A_{1,i}(r|\omega) = 1$. As above, we can conclude that $A_{1,i}(r|\omega) = 1$ implies that $r$ will be offered by $i$ in all TPEs for state $\omega$. On the other hand, if $A_{2,i}(r|\omega) = 0$, then $A_{\sigma^*,i}(r|\omega) = 0$ is claimed to be implied. Player $i$ will offer his participation in $r$ only with minimal probability (under full support) if $v_i(E'(r)) < v_i(r^o)$, which applies if $A_{2,i}(r|\omega) = 0$. Hence, if $A_{2,i}(r|\omega) = 0$, then he will not offer his participation in any TPE. This holds for all $i$, and thus completes the proof. □