VIII.3 Generalizations

Case: Finite number of strategies, finite number of players:
Straight forward, but no longer matrix representation

Nash's Theorem:
Every game with a finite number of players and a finite number of
strategies for every player has at least one equilibrium in mixed
strategies.

Case: Finite number of players, infinite number of strategies
Example 1: Cournot Oligopoly, homogeneous good
Suppliers \(i = 1, \ldots, n\); marginal costs \(c_1, \ldots, c_n\)
strategies \(x_i = \) quantities of a homogeneous good
market price \(p = 1 - x, x = x_1 + \ldots + x_n\)
⇒ profit of \(i \) is \(G_i = x_i(p-c_i)\)
Best reply to \(x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\)?
\[
\frac{\partial G_i}{\partial x_j} = p - c_i + x_j \frac{\partial p}{\partial x_j} = -1
\]
\[
= 1 - c_i - x_1 - \ldots - x_{i-1} - 2x_i - x_{i+1} - \ldots - x_n
\]
\[
(*) \quad \frac{\partial^2 G_i}{\partial x_j^2} = -2
\]
\[
(**) \quad x_i = \frac{1 - c_i - x_1 - \ldots - x_{i-1} - x_{i+1} - \ldots - x_n}{2}
\]
is the best reply function if this \(x_i\) is non-negative, otherwise
\[
x_i = 0
\]
because of (*).
In a Nash equilibrium all \(x_i\) have to be best replies to one another. Ie, in
the case of interior maximum \(\frac{\partial G_i}{\partial x_i} = 0\) must apply.

From the latter we get
\[
x_1 + \ldots + x_{i-1} + 2x_i + x_{i+1} + \ldots + x_n = x + x_i = 1 - c_i, \quad i = 1, \ldots, n
\]
With
\[
x = x_1 + \ldots + x_i + \ldots + x_n
\]
Adding up all these equations delivers
\[
(n + 1)x = n - (c_1 + \ldots + c_i + \ldots + c_n)
\]
\[
x = \frac{n - (c_1 + \ldots + c_i + \ldots + c_n)}{n + 1}
\]
\[
x_i = 1 - c_i - x, \quad i = 1, \ldots, n
\]
\[
p = 1 - x = \frac{1 + c_1 + \ldots + c_i + \ldots + c_n}{n + 1}
\]
An interior solution requires \(c_n < p\). Show that this implies that \(p\) decreases
with \(n\).
Example 2: Cournot Oligopoly, inhomogeneous goods

Suppliers $i = 1, \ldots, n$; marginal costs $c_i$, ..., $c_n$

strategies $x_i = $ quantities of good $i$

market prices $p_i = 1 - x_i \cdot \sum_{k \neq i} \alpha_k x_k$, $\alpha_k < 1$

$\alpha_k > 0$ for substitutes, $\alpha_k < 0$ for complements.

Example 3: Bertrand Oligopoly, inhomogeneous goods

Suppliers $i = 1, \ldots, n$; marginal costs $c_i$, ..., $c_n$

strategies $p_i = $ price of good $i$

demand $x_i = 1 - p_i + \sum_{k \neq i} \alpha_k p_k$, $\alpha_k < 1$

$\alpha_k > 0$ for substitutes, $\alpha_k < 0$ for complements.

Experimental results

Number of competitors (homogeneous goods)

Results by Huck, Normann, Oechssler (and others):

\[ r = \frac{\overline{Q}}{Q^n} \]

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<th>number of firms</th>
<th>$Q^n$</th>
<th>$Q_{1-25}$</th>
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<td>(6.68)</td>
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<td>(5.56)</td>
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Note: Standard errors in parenthesis.

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<th>rounds</th>
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<th>$r$</th>
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Notes: *random end, ** asymmetric costs, HNO = Huck, Normann, Oechssler, IMN = Huck, Müller Normann.
Further results of a meta-study by Engel (2007) emphasize that there are many interaction effects, i.e. often it is not possible to linearly disentangle the effects of different variables:

- competition with (in-)homogenous goods and (a-)symmetric players yield lower prices and profits (-) or higher prices and profits (+) than the Nash equilibrium, according to

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<td>inhomogenous</td>
<td>+</td>
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- competition with (in-)homogenous goods and players who are re-matched every round (partners) or get new co-players in every round (strangers) yield lower prices and profits (-) or higher prices and profits (+) than the Nash equilibrium, according to

<table>
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</table>

Further interaction effects: With two competitors, partners collude more successfully than strangers, with more than two, strangers collude more than partners. This result changes if feedback information (about behaviour in previous rounds) is given also for strangers.

With infinitely many strategies, does an equilibrium always exist?

**Counter-examples:**  Jüdisches Poker (short story by Ephraim Kishon)

Bertrand-Duopoly with different constant marginal costs

In the preface, we gave the reader a light whiff of the chauvinistic atmosphere prevailing in Israel, which cannot be condemned in too strong words. To give him further proof of our specifically Jewish mentality, we think we could not do better than describe the game of poker we had with our friend Ervinke one sleepy afternoon.

This ought to bring the reader nearer to understanding the Jewish soul than anything NBC’s most skilled Middle East commentator could tell him.

**Jewish Poker**

Far quite a while the two of us sat at our table, wordlessly stirring our coffee. Ervinke was bored. "All right," he said. "Let’s play poker.”


He then briefly explained the rules of the game. Jewish poker is played without cards, in your head, as befits the People of the Book.

“You think of a number, I also think of a number,” Ervinke said. “Whoever thinks of a higher number wins. This sounds easy, but it has a hundred pitfalls. Nu!”

“All right,” I agreed. “Let’s try.”

We plunked down five piasters each, and. Leaning back in our chairs began to think of numbers. After a while Ervinke signaled that he had one. I said I was ready.

“All right,” thus Ervinke. “Let’s hear your number.”

“Eleven,” I said.

“Twelve,” Ervinke said, and took the money. I could have kicked myself, because originally I had thought of Fourteen, and only at the last moment had I climbed down to Eleven, I really don’t know why.
“Listen.” I turned to Ervinke. “What would have happened had I said Fourteen?”
“What a question! I’d have lost. Now, that is just the charm of poker: you never
know how things will turn out. But if your nerves cannot stand a little gambling,
perhaps we had better call it off.”
Without saying another word, I put down ten piasters on the table. Ervinke did
likewise. I pondered my number carefully and opened with Eighteen.
“Damn!” Ervinke said. “I have only Seventeen!” I swept the money into my
pocket and quietly guffawed. Ervinke had certainly not dreamed that I would
master the tricks of Jewish poker so quickly. He had probably counted on my
opening with Fifteen or Sixteen, but certainly not with Eighteen. Ervinke, his
brow in angry furrows, proposed the double the stakes.
“As you like,” I sneered, and could hardly keep back my jubilant laughter. In the
meantime a fantastic number had occurred to me: Thirty-five!
“Lead!” said Ervinke. “Thirty-five!”
“Forty-three!”
With that he pocketed the forty piasters. I could feel the blood rushing into my brain.
“Listen,” I hissed. “Then why didn’t you say Forty-three the last time?”
“Because I had thought of Seventeen!” Ervinke retorted indignantly. “Don’t you
see, that is the fun in poker: you never know what will happen next.”
“A pound,” I remarked dryly, and, my lips curled in scorn, I threw a note on the
table. Ervinke extracted a similar note from his pocket and with maddening
slowness placed it next to mine. The tension was unbearable. I opened with
Fifty-four.
“Oh, damn it!” Ervinke fumed. “I also thought of Fifty-four! Draw! Another
game!”
My brain worked with lightning speed. “Now you think I’ll again call Eleven, my
boy,” I reasoned. “But you’ll get the surprise of your life.” I chose the sure-fire Sixty-nine.
“You know what, Ervinke”- I turned to Ervinke - “you lead.”
“As you like,” he agreed. “It’s all the same with me. Seventy!”
Everything went black before my eyes. I had not felt such panic since the siege of
Jerusalem.
“Nu?” Ervinke urged. “What number did you think of?”
“What do you know?” I whispered with downcast eyes. “I have forgotten.”
“You liar!” Ervinke flared up. “I know you didn’t forget, but simply thought of a
smaller number and now don’t want to own up. An old trick. Shame on you!”
I almost slapped his loathsome face for this evil slander, but with some difficulty
overcame the urge. With blazing eyes I upped the stakes by another pound and
thought of a murderous number: Ninety-six!
“Lead, stinker,” I threw at Ervinke, whereupon he leaned across the table and
hissed into my face: “Sixteen hundred and eighty-three!”
A queer weakness gripped me.
“Eighteen hundred,” I mumbled warily. “Double!” Ervinke shouted, and
pocketed the four pounds.
“What do you mean, ‘double’?” I snorted. “What’s that?”
“If you lose your temper in poker, you’ll lose your shirt!” Ervinke lectured me.
“Any child will understand that my number doubled is higher than yours, so it’s
clear that…”
“Two thousand four hundred and seventeen,” thus Ervinke.
“Double!” I sneered, and grabbed the stakes, but Ervinke caught my hand.
“Redouble!” he whispered, and pocketed the tenner. I felt I was going out of my
mind.
“Listen” – I gritted my teeth – “if that’s how things stand, I could also have said
‘redouble’ in the last game, couldn’t I?”
“To tell you the truth, I was rather surprised that
you didn’t. But this is poker,
3
you either know how to play it or you
don’t! If you are scatterbrained, better stick to croquet.”
The stakes were ten pounds. “Lead!” I screamed. Ervinke leaned back in his
chair, and in a disquietingly calm voice announced his number: Four.
“Ten million!” I blared triumphantly. But without the slightest sign of excitement,
Ervinke said: “Ultimo!”
And took the twenty pounds.
I then broke into sobs. Ervinke stroked my hair and told me that according to Hoyle,
whoever is first out with the ultimo wins, regardless of numbers. That is the fun in
poker: you have to make split-second decisions.
“Twenty pounds,” I whimpered, and placed my last notes in the hands of fate.
Ervinke also placed his money. My face was bathed in cold sweat. Ervinke
went on calmly blowing smoke rings, only his eyes had narrowed.

"Who leads?"

"You," I answered, and he fell into my trap like the sucker he was.

"So I lead," Ervinke said. "Ultimo", and he stretched out his hand for the treasure.

"Just a moment" – I stopped him – "Ben-Gurion!"

With that I pocketed the Mint's six-month output.

"Ben-Gurion is even stronger than ultimo," I explained. "But it's getting dark outside. Perhaps we had better break it off."

We paid the waiter and left.

Ervinke asked for his money back, saying that I had invented the Ben-Gurion on the spur of the moment. I admitted this, but said that the fun in poker was just in the rule that you never returned the money you had won.

\[\text{Ervinke is not a Jewish name at all. To spare the reader the trouble of remembering tongue-twisting genuine Hebrew names, we decided to use names more familiar to the average civilized person. Ervinke's genuine Israel name would be Elyakim. Or take the present author: he started out in life as Francis, but is now better known as Ephraim.}\\]

\[\text{The interjection "Nu!"-somewhat like the English "Well!"-plays the role of Jolly Joker in the Hebrew language. According to a superficial estimate, it has about 680 meanings, depending upon the speaker's facial expression and the time of day. Here are a few meanings Picked at random:}\\

"Come on!"

"Please, leave me alone, can't you?"

"I didn't understand a word of what you said. What on earth are you driving at?"

"All right, suppose things are as you say, though mind you, I don't say so. Is that reason enough to start shouting as if I had trod on your corns? Bloody fool! Yes, you. Really... ."

\[\text{Yahabibi - Arab equivalent of "old chap." This address is used only by very intimate friends, or in the street by complete strangers, or school children at school, or Cabinet Ministers among themselves.}\]