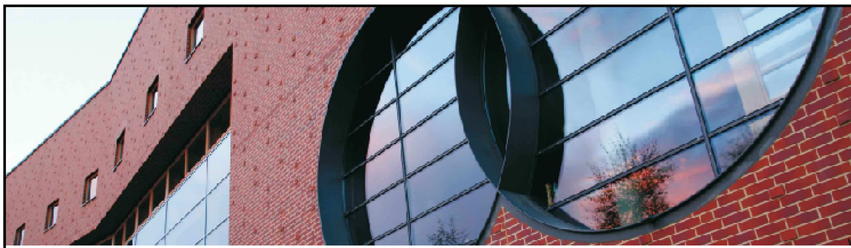


Claudia Vogel

Mathematics for IBA

Winter Term 2009/2010

EUROPA-UNIVERSITÄT VIADRINA FRANKFURT (ODER)

Outline

5. Game Theory
   - **Introduction & General Techniques**
   - Sequential Move Games
   - Simultaneous Move Games with Pure Strategies
   - Combining Sequential & Simultaneous Moves
   - Simultaneous Move Games with Mixed Strategies
   - Discussion

Motivation: Why study Game Theory?

- Games are played in many situation of every days life
  - Roomates and Families
  - Professors and Students
  - Dating

- Other fields of application
  - Politics, Economics, Business
  - Conflict Resolution
  - Evolutionary Biology
  - Sports

The beginnings of Game Theory

1944 “Theory of Games and Economic Behavior“

Oskar Morgenstern & John Neumann

Decisions vs Games

- Decision: a situation in which a person chooses from different alternatives without concerning reactions from others
- Game: interaction between mutually aware players
  - Mutual awareness: The actions of person A affect person B, B knows this and reacts or takes advance actions; A includes this into his decision process

Sequential vs Simultaneous Move Games

- Sequential Move Game: player move one after the other
  - Example: chess
- Simultaneous Move Game: Players act at the same time and without knowing, what action the other player chose
  - Example: race to develop a new medicine

Conflict in Players‘ Interests

- Zero Sum Game: one player’s gain is the other player’s loss
  - Total available gain: Zero
  - Complete conflict of players’ interests
- Constant Sum Game: The total available gain is not exactly zero, but constant.
- Games in trade or other economic activities usually offer benefits for everyone and are not zero-sum.
  - Example: Joint ventures

One-shot vs Repeated Games

- One-shot Game: only one interaction between the players
  - No knowledge about the opponent
- Repeated Game: repeated interactions
  - With the same opponent: Reputation
    Example: Long-Term Business Relations
  - With changing opponents: information about usual behavior
    Example: Price setting in the Turkish Bazaar

Information

- Perfect information: all information is available to all players
  - Example: chess
- Imperfect Information
  - External uncertainty: uncertainty about relevant variables (e.g. weather)
  - Strategic uncertainty: about opponents past moves
- Asymmetric Information: some information is only available to one player
  - Example: job market

Cooperative vs Non-Cooperative Games

- Cooperative Game: players are able to enforce contracts
  - Example: The European Union
- Non-cooperative Game: cooperation has to be self-enforcing as it is not enforceable by an outside party
  - Example: The European Union and non-member countries

Terminology

- Strategy: (choices) action available to the players
- Payoff: the number associated with each possible outcome for each player
  - Expected Payoff: probability weighted average payoff
- Rationality: A player has a consistent set of payoffs over all possible outcomes and calculates the strategy that best serves his interests
- Equilibrium: Each player is using the strategy that is the best response to the strategies of the other players

Dynamics and Evolutionary Games

- Evolutionary Games allow dynamic processes in which strategies that proved to be better are more likely to be chosen in future games.
- Each player has a particular "programmed" strategy and meets players with the same or other "programmed" strategies → better strategies multiply faster; worse strategies decline
- Questions for the analysis: Does the dynamic process converge to an evolutionary stable state? Does just one strategy survive in the end or can several strategies coexist?

Observation and Experiment

- Theory always should relate to reality in two ways:
  – Reality should help structure theory.
  – Reality should provide a check on the results of theory.

- Reality of strategic interactions can be found out by
  – Observation
  – Special experiments

Introduction to Probabilities

- Probability: The probability of a random event is a quantitative measure of the likelihood of ist occurence. For events that can be observed in repeated trials, it is the long-run frequency with which it occurs. For unique events or other situations where uncertainty may be in the mind of a person, other measures are constructd, such as subjective probability.
- Addition Rule: If you devide a set of events into a number of subsets, none of which overlap, then the probabilities of each subset occuring must sum to the probability of the full set of events; if that full set includes all possible outcomes, then ist probability is 1.
- Expected value: The probability-weighted average of the outcomes of a random variable, that is, ist statistical mean or expectation.

Summary

- Strategic game situations are distinguished from individual decision-making situation by the presence of significant interactions among the players.
- Players have strategies that lead to different outcomes with different associated payoffs.
- Payoffs incorporate everything that is important to a player and are calculated by using probalistic averages or expectations if outcomes are random and include some risk.
- Game Theory may be used for explanation, prediction, or prescription in various circumstances

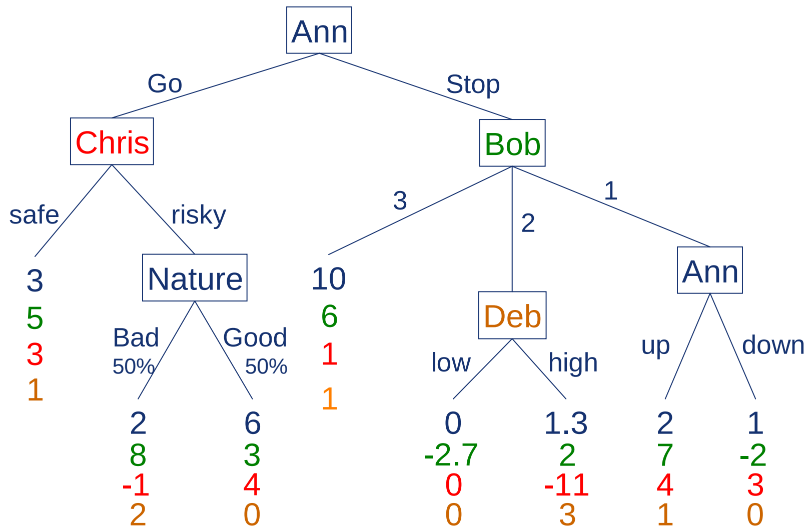
Outline

5. Game Theory
   - Introduction & General Techniques
   - **Sequential Move Games**
   - Simultaneous Move Games with Pure Strategies
   - Combining Sequential & Simultaneous Moves
   - Simultaneous Move Games with Mixed Strategies
   - Discussion

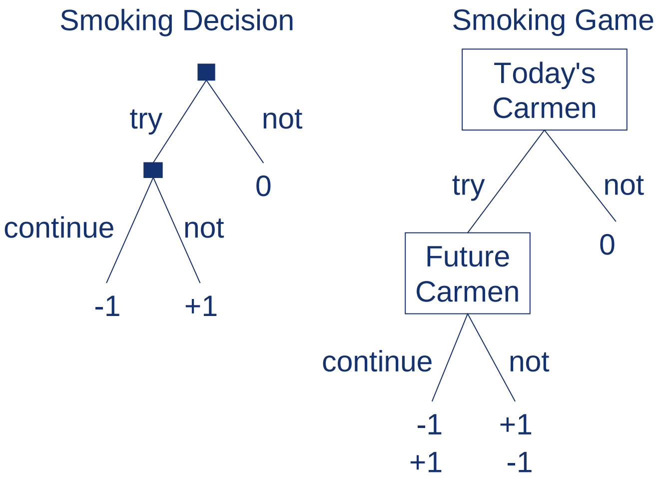
Game Tree 1/3

- Game Tree: Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.
- Node: is a point from which branches emerge, or where a branch terminates.
- Branch: Each branch emerging from a node represents one action that can be taken at that node.
- Terminal node: represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.
- Move: an action at one node of a game tree.
- Strategy: A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is turn to act.

Game Tree 2/3



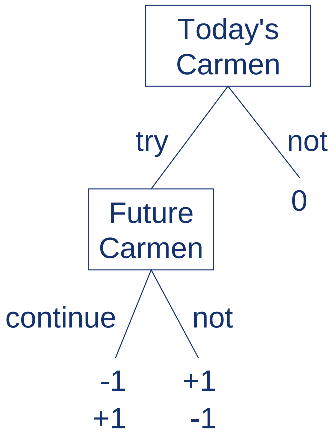
Game Tree 3/3



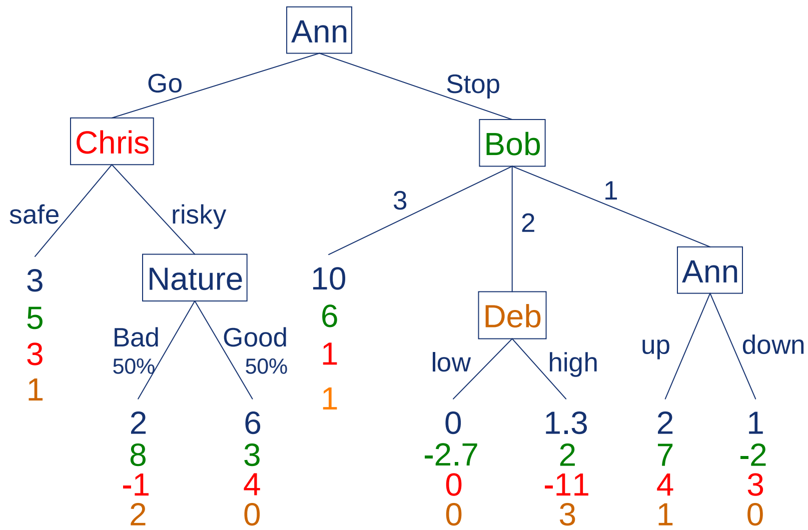
Solving Game Trees 1/3

- Rollback: Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal node and working backward to the initial node.
  - prune: identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played
- Rollback equilibrium: The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

Solving Game Trees 2/3



Solving Game Trees 3/3



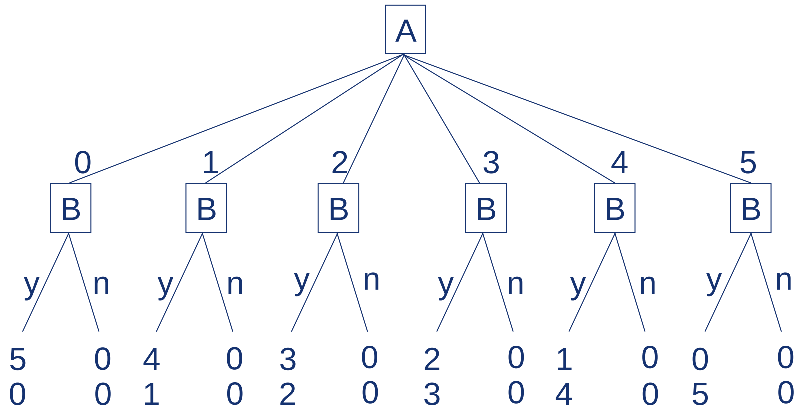
Order Advantages

- First-mover advantage: it is beneficial to set the stage for a later player
- Second-mover advantage: beneficial to react to the moves of the first player
  - Example: price setting
- In some games the result is determined by the setup of the game and the order of moves doesn't matter
  - Example: removing matchsticks from a pile

Experimental Results

- **Ultimatum Bargaining:** Player A has the possibility to divide an amount of 5 Euro between himself and player B. After the proposal of the allocation, player B decides whether to accept or not.
  - If player B accepts both receive the money according to the offer.
  - If player B does not accept, both get nothing.
- Assuming that player A can only offer in steps of 1 Euro coins, he can either offer 0,1,2,3,4 or 5 Euro.

Game Tree Ultimatum Bargaining



Summary

- Sequential move games require players to consider the future consequences of their current moves before choosing their actions.
- Game trees illustrate the games. They consist of nodes and branches that show all possible actions of each player and the associated outcomes.
- The rollback equilibrium is a complete plan of actions of all players.
- Different types of games entail advantages for different players, e.g. first-mover advantages.

Outline

5. Game Theory
   - Introduction & General Techniques
   - Sequential Move Games
   - **Simultaneous Move Games with Pure Strategies**
   - Combining Sequential & Simultaneous Moves
   - Simultaneous Move Games with Mixed Strategies
   - Discussion

Simultaneous Move Games

- Players choose their actions at exactly the same time.
- No information about what the other player has done or what he will do.
- Also called games of imperfect information or imperfect knowledge.
- Many situation can be described by such games: producers in an industry usually make their decisions about product design and features without knowing, what their competitors are doing, voters in elections vote simultaneously.
- Strategies cannot be made contingent on the other player's action, the player can only reason through the game from the perspective of the opponent.

Illustration of Simultaneous Move Games

- Game matrix: A table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column); and each cell shows the payoffs of all players in a specified order, corresponding to the configuration of strategies which yield that cell. Also called game table or payoff table.
- Game matrix is called the normal form or strategic form of a simultaneous move game.
- Pure Strategies: actions that are available to each player.
- Mixed Strategy: a random choice to be made with the specified probabilities, from a player's originally specified pure strategies.

Example of a Game Table

		Column		
		Left	Middle	Right
Row	Top	3,1	2,3	10,2
	High	4,5	3,0	6,4
	Low	2,2	5,4	12,3
	Bottom	5,6	4,5	9,7

Nash equilibrium

- Nash equilibrium: A configuration of strategies (one for each player) such that each player's strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)
- No player should want to change his strategy once he has seen what his rivals did, i.e. each action chose is the best response to the other players' actions.

John Nash, mathematician, economist, Nobel Prize winner

Finding the Nash equilibrium

		Column		
		Left	Middle	Right
Row	Top	3,1	2,3	10,2
	High	4,5	3,0	6,4
	Low	2,2	5,4	12,3
	Bottom	5,6	4,5	9,7

Dominance

- Dominant strategy: A strategy X is dominant if, for each permissible strategy configuration of the other player, X gives him a higher payoff than any of his other strategies. That is, his best response is always X.
- Dominated strategy: A strategy Y is dominated for a player if there is another strategy X such that, X always gives a higher payoff.
- Successive (or iterated) elimination of dominated strategies: Considering the players in turns and repeating the process in rotation, eliminating all strategies that are dominated for one at a time, and continuing doing so until no such further elimination is possible.
- If the process yields a unique outcome the game is dominance solvable.

The Prisoners' Dilemma 1/2

Consider a story line from a typical television crime program: A husband and wife are under suspicion that they were conspirators in a murder. The detectives place them in separate detention rooms and interrogate them one at a time. There is little concrete evidence linking the pair to murder, although there is some evidence they were involved in kidnapping the victim. The detectives explain to each suspect that they are both looking at jail time for the kidnapping charge (3 years) even if there is no confession from either of them. In addition they are told that the detectives "know" how one had been coerced by the other to participate in the crime; it is implied that jail time for a solitary confessor will be significantly reduced (to only 1year instead of 25 for murder) if the whole story is committed. If both confess, jail terms could be negotiated down (10 years) but not as much as in one confession and one denial.

The Prisoners' Dilemma 2/2

		Wife	
		CONFESS (Defect)	DENY (Cooperate)
Husband	CONFESS (Defect)	10yr, 10yr	1yr, 25yr
	DENY (Cooperate)	25yr, 1yr	3yr, 3yr

One player Dominance

		Federal Reserve	
		Low interest rates	High interest rates
Congress	Budget Balance	3, 4	1, 3
	Budget Deficit	4, 1	2, 2

Weak Dominance

		Player 2	
		a	b
Player 1	x	1,1	1,0
	y	0,1	1,1

Classes of Games

- one Nash-equilibrium in pure strategies
  - Prisoner's Dilemma
- multiple Nash-equilibria in pure strategies
  - Pure coordination
  - Assurance
  - Battle of Sexes
  - Chicken
- no equilibrium in pure strategies

The Prisoners' Dilemma

		Wife	
		CONFESS (Defect)	DENY (Cooperate)
Husband	CONFESS (Defect)	10yr, 10yr	1yr, 25yr
	DENY (Cooperate)	25yr, 1yr	3yr, 3yr

Each player has two strategies, Cooperate and Defect, such that for each player Defect dominates Cooperate and the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate)

Pure coordination

		Sally	
		Starbucks	Local Latte
Harry	Starbucks	1,1	0,0
	Local Latte	0,0	1,1

All players are indifferent among all Nash equilibria, and coordination is needed only to ensure avoidance of a non-equilibrium outcome.

Assurance

		Sally	
		Starbucks	Local Latte
Harry	Starbucks	1,1	0,0
	Local Latte	0,0	2,2

All players prefer the outcome at Local Latte, as this leads to higher payoffs. This equilibrium can act as a Focal Point.

Battle of Sexes

		Sally	
		Starbucks	Local Latte
Harry	Starbucks	2,1	0,0
	Local Latte	0,0	1,2

Each player has a Hard and a Soft strategy. Each player prefers the outcome where he is Hard and the other is Soft, but both prefer the Nash equilibria to the other two possibilities

Chicken

		Dean	
		Swerve (Chicken)	Straight (Tough)
James	Swerve (Chicken)	0, 0	-1, 1
	Straight (Tough)	1, -1	-2, -2

Each player has a tough and a weak strategy. Each player prefers the strategy where he is tough and the other is weak, but the outcome (tough, tough) is worst for both.

No equilibrium in pure strategies

		Navratilova	
		DL	CC
Evert	DL	50,50	80,20
	CC	90,10	20,80

Summary

- Simultaneous move games differ from sequential move games in that players make decisions without knowing their rivals' actions.
- The games are illustrated in game tables.
- The solution is the Nash equilibrium, which exists when each player chooses that strategy that is best for him, given that all other players are using their equilibrium strategies.
- Nash equilibria can be found by successive elimination of dominated strategies or cell-by-cell inspection
- Specific Games include Prisoners' Dilemma, Coordination games such as assurance, chicken and battle of sexes and matching pennis.

Outline

5. Game Theory
    - Introduction & General Techniques
    - Sequential Move Games
    - Simultaneous Move Games with Pure Strategies
    - **Combining Sequential & Simultaneous Moves**
    - Simultaneous Move Games with Mixed Strategies
    - Discussion

Changing the order of moves in a game

- Sequential-move games become simultaneous if the players cannot observe moves made by their rivals
  - Analysis: search for Nash equilibrium rather than rollback!
- Simultaneous-move games become sequential if one player is able to observe the other's move before choosing her own
- Any changes to the rules of the game can change its outcome

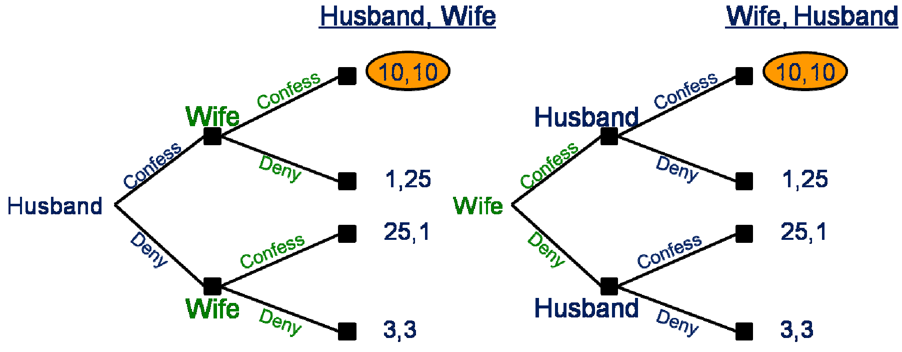
Example 1: No change in outcome 1/2

		Wife	
		CONFESS (Defect)	DENY (Cooperate)
Husband	CONFESS (Defect)	10yr, 10yr	1yr, 25yr
	DENY (Cooperate)	25yr, 1yr	3yr, 3yr

Example 1: No change in outcome 2/2

Sequential play:

Husband moves first | Wife moves first



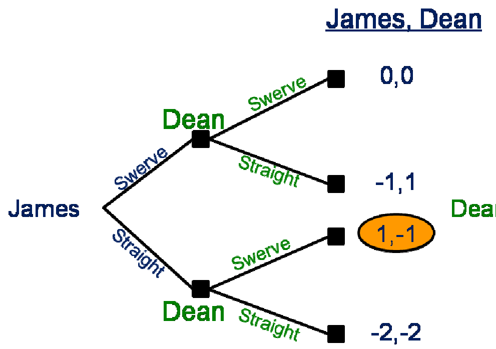
Example 2: First-mover advantage 1/2

		Dean: Swerve (Chicken)	Dean: Straight (Tough)
James	Swerve (Chicken)	0, 0	-1, 1
	Straight (Tough)	1, -1	-2, -2

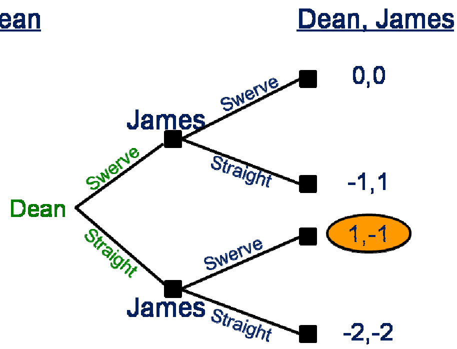
Example 2: First-mover advantage 2/2

Sequential play:

James moves first



Dean moves first



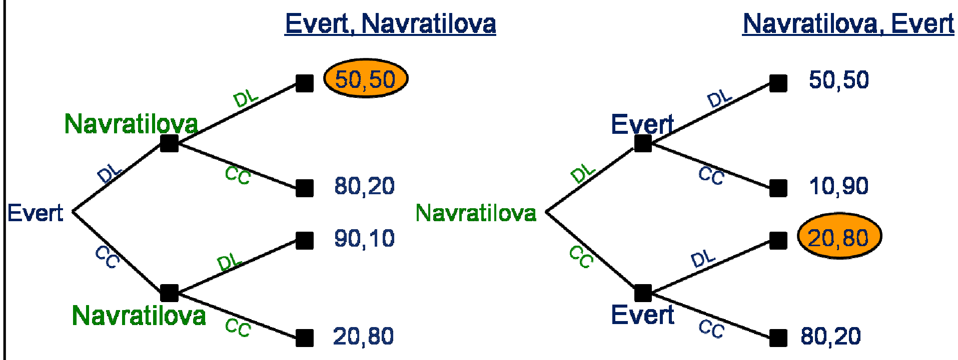
Example 3: Second-mover advantage 1/2

		Navratilova	
		DL	CC
Evert	DL	50,50	80,20
	CC	90,10	20,80

Example 3: Second-mover advantage 2/2

Sequential play:

Evert moves first — Navratilova moves first



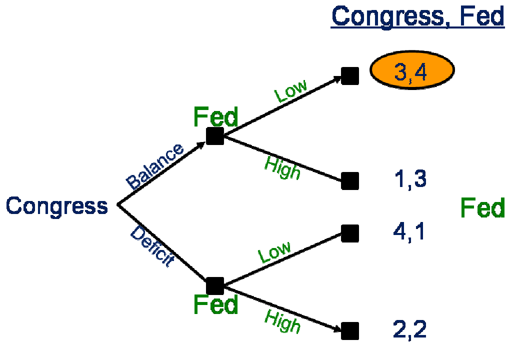
Example 4: Both players do better 1/2

		Federal Reserve	
		Low interest rates	High interest rates
Congress	Budget Balance	3, 4	1, 3
	Budget Deficit	4, 1	2, 2

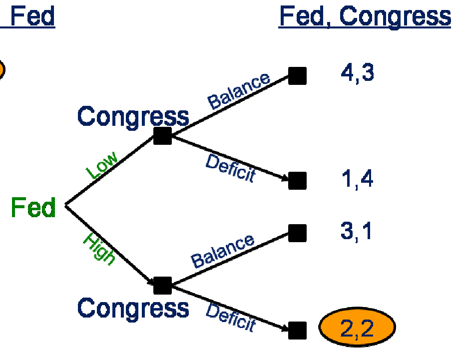
Example 4: Both players do better 2/2

Sequential play:

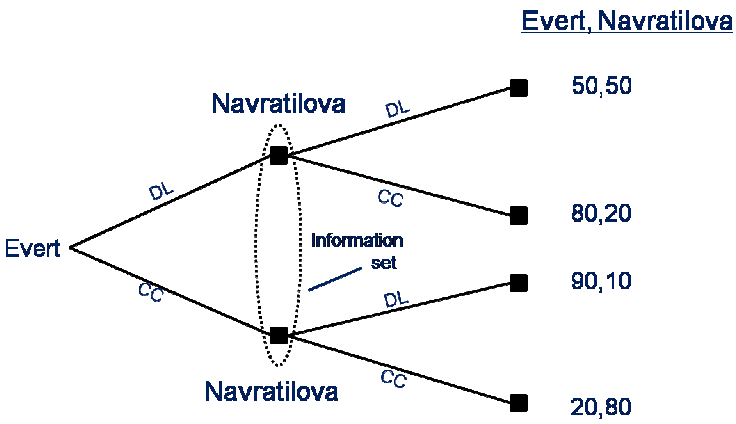
Congress moves first



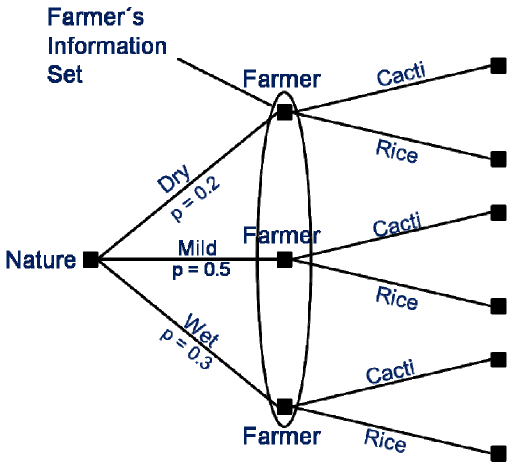
Fed moves first



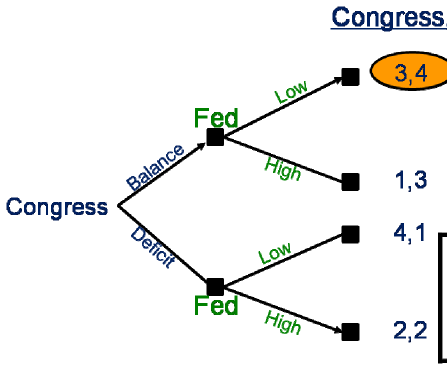
Illustrating simultaneous-move games by using game trees



External Uncertainty



Analyzing sequential-move games in strategic form

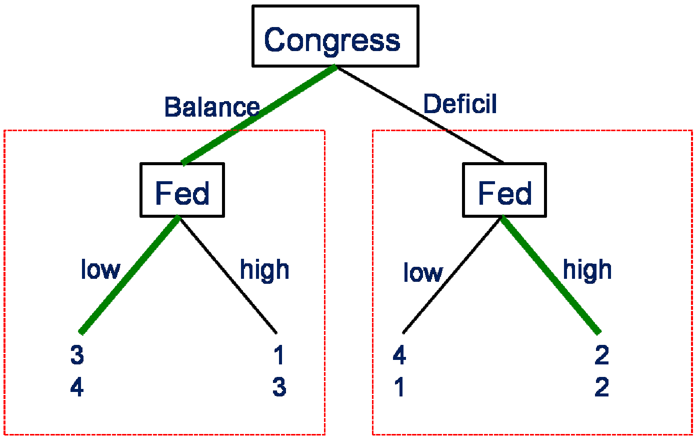


		Federal Reserve			
		Low if B High if D	High if B Low if D	Low, Low	High, High
Congress	Balance	3,4	1,3	3,4	1,3
	Deficit	2,2	4,1	4,1	2,2

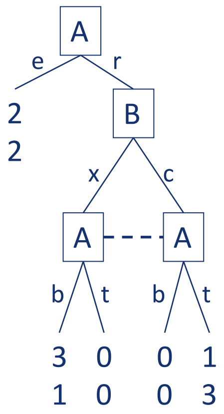
Further Definitions 1/2

- Subgame: A portion of a larger game, starting at a non initial node of the larger game
- Off-equilibrium path: a path of play that does not result from the players´ choices of strategies
- Off-equilibrium subgame: a subgame starting at a node, that does not lie on the equilibrium path of play
- Continuation: the remaining part of the plan of action of that strategy, applicable to the subgame that starts at this node
- Subgame-perfect equilibrium: A configuration of strategies (complete plans of action) such that their continuation in any subgame remains optimal (part of a roll back equilibrium), whether that subagme is on- or off-equilibrium.

Further Definitions 2/2



Games with sequential and simultaneous moves



Summary

- When illustrating simultaneous games with a game tree information sets contain those nodes in which players do not have information about previous actions.
- Changing the rules of a game to alter the timing of actions can alter the equilibrium outcome of the game:
  - A sequential move game has one rollback equilibrium, but the same game played simultaneously may have several Nash equilibria.
  - A sequential game analyzed from its strategic form may have many possible Nash equilibria.
  - By using the criterion of credibility, one can eliminate some strategies; this leads to the subgame-perfect equilibrium of the sequential game.

Outline

5. Game Theory
   - Introduction & General Techniques
   - Sequential Move Games
   - Simultaneous Move Games with Pure Strategies
   - Combining Sequential & Simultaneous Moves
   - **Simultaneous Move Games with Mixed Strategies**
   - Discussion

Mixed Strategies

- pure strategy: nonrandom plan of action for each player
- mixed strategy: the actual move is chosen randomly from the set of pure strategies with some specific probabilities
- Under general conditions every simultaneous move game has a Nash equilibrium in mixed strategies.

Example

		Player 1		
		x	y	z
Player 2	a	4,10	1,0	1,3
	b	7,0	0,10	10,3

Best responses 1/3

		Player 1	
		p x	y **(1-p)**
Player 2	a **q**	4,10	1,0
	b **(1-q)**	7,0	0,10

Expected value of player 1:

E_1=10pq +0(1-p)q +0p(1-q) +10(1-p)(1-q) =20pq-10p-10q+10

Expected value of player 2:

E =4pq+1(1-p)q+7p(1-q) +0(1-p)(1-q) = - 4pq+7p+q

Best responses 2/3

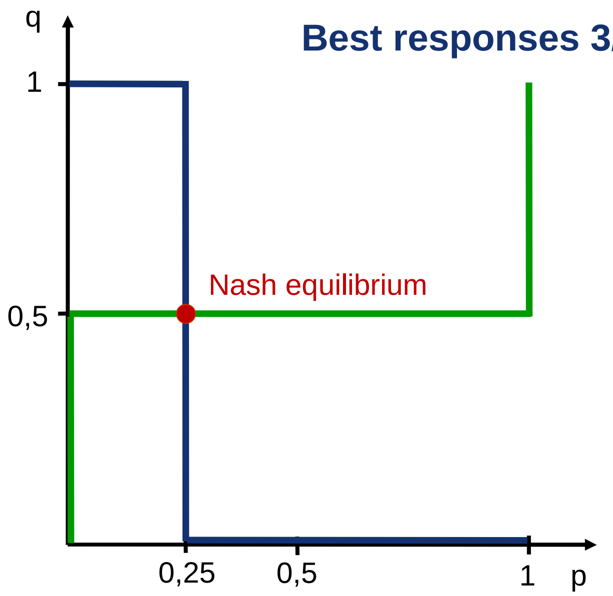
Best response of player 1 to q of player 2:
$E_1=p(20q-10)-10q+10$

q>0.5	(20q-10)>0	best response:	p=1
q=0.5	(20q-10)=0	best response:	p any
q<0.5	(20q-10)<0	best response:	p=0

Best response of player 2 to p of player 1:
$E\ =q(-4p+1)+7p$

p<0.25	(- 4p+1)>0	best response:	q=1
p=0.25	(- 4p+1)=0	best response:	q any
p>0.25	(- 4p+1)<0	best response:	q=0

Best responses 3/3



Mixed Strategies in Chicken

		Dean	
		swerve **p**	straight **(1-p)**
James	swerve **q**	0,0	-1,1
	straight **(1-q)**	1,-1	-2,-2

Expected value of DEAN:

E_D=0pq +1(1-p)q -1p(1-q) -2(1-p)(1-q) =p(1-2q)+3p-2

Expected value of JAMES:

E =0pq -1(1-p)q +1p(1-q) -2(1-p)(1-q) = q(1-2p)+3p-2

Best responses 1/2

Best response of DEAN to q of JAMES:

E_D= p(1-2q)+3p-2

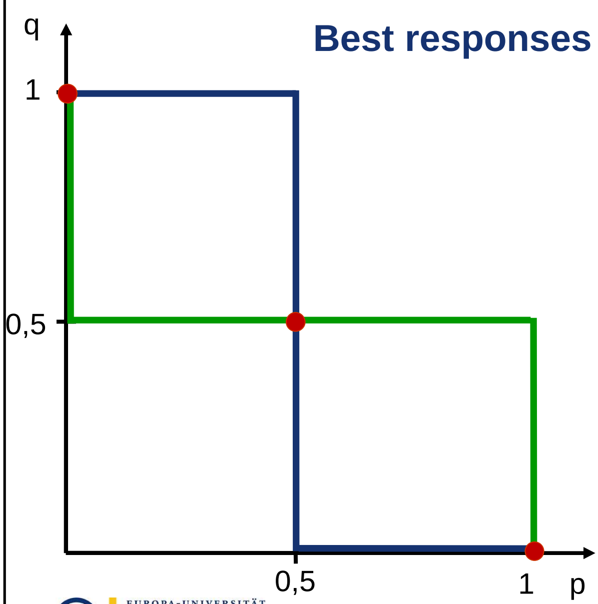
q<0.5	(1-2q)>0	best response:	p=1
q=0.5	(1-2q)=0	best response:	p any
q>0.5	(1-2q)<0	best response:	p=0

Best response of JAMES to p of DEAN:

E =q(1-2p)+3p-2

p<0.5	(1-2q)>0	best response:	q=1
p=0.5	(1-2q)=0	best response:	q any
p>0.5	(1-2q)<0	best response:	q=0

Best responses 2/2



General case

		Player 1	
		p x	y **(1-p)**
Player 2	a **q**	α,a	β,b
	b **(1-q)**	γ,c	δ,d

Expected value of player 1:

E_1=apq+b(1-p)q+cp(1-q)+d(1-p)(1-q)

=p[qa-qb+(1-q)c-(1-q)d]+qb+(1-q)d

Best response of player 1 for (q,1-q) of player 2

E =p[qa-qb+(1-q)c-(1-q)d]+qb+(1-q)d

$$p = \begin{cases} 1 & \text{for } [\ldots]>0 \\ \text{any} & \text{for } [\ldots]=0 \\ 0 & \text{for } [\ldots]<0 \end{cases}$$

Analog process for player 2.

Indifference Criteria

A mixed strategy equilibrium requests for both players to be indifferent between their strategies ➔ […]=0

Solving […]=0 for p respectively q results in the formulas

$$p = \frac{\delta-\beta}{\alpha-\beta-\gamma+\delta} \qquad q = \frac{d-c}{a-b-c+d}$$

Summary

- If players want to be unpredictable, they do so by using a mixed strategy that specifies a probability distribution over their set of available pure strategies.
- Every simultaneous move game where each player has a finite number of pure strategies has a Nash equilibrium in mixed strategies.
- Equilibrium mixtures for two players must be best responses to each other.
- Graphs of best-response curves for both players show all equilibria of the game at the intersection(s) of the curves.
- Good use of mixed strategy requires that there be no predictable system of randomization.

Outline

5. Game Theory
    - Introduction & General Techniques
    - Sequential Move Games
    - Simultaneous Move Games with Pure Strategies
    - Combining Sequential & Simultaneous Moves
    - Simultaneous Move Games with Mixed Strategies
    - **Discussion**

Empirical evidence concerning Nash equilibrium

Do people actually play Nash equilibrium strategies in experiments?

•Yes, in simple, single-move games with a unique Nash equilibrium after some repetition

•Mixed evidence, for

- more complex or repeated situations
- coordination among multiple equilibria
- games with complex calculations of the Nash equilibrium

Critical Discussion of the Nash equilibrium concept 1/3

		Column A	Column B	Column C
Row	A	2,2	3,1	0,2
Row	B	1,3	2,2	3,2
Row	C	2,0	2,3	2,2

Critical Discussion of the Nash equilibrium concept 2/3

		B	
		Left	Right
A	Up	9,10	8,9.9
	Down	10,10	-1000,9.9

Critical Discussion of the Nash equilibrium concept 3/3

1. The Nash equilibrium is too imprecise.
2. Players in actual games do not play Nash equilibrium strategies.
3. Rationality does not itself imply Nash equilibrium.

➔ Have considerable confidence in the Nash equilibrium concept if it is played frequently by players from a reasonably stable population and under relatively unchanging rules and conditions.
➔ Be cautious if the game is new, played just once and players are inexperienced.
➔ Treat games that are played repeatedly by the same group of players as a game in its own right.