The Envelope Paradox, the Siegel Paradox, and the Impossibility of Random Walks in Equity and Financial Markets

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Abstract

It is shown that the Siegel Paradox is very close to the Envelope Paradox. While the latter is understood, i.e. reduced to the unveiling of hidden assumptions (Nalebuff, 1989), there is an ongoing discussion about the significance of the further (from Siegel, 1972, to Kemp and Sinn, 2000), in particular about its significance for currency speculation. The comparison of the paradoxes help to unveil the hidden assumptions in the Siegel Paradox.

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I. The two paradoxes

In an article in The Journal of Economic Perspectives, Nalebuff (1989) discussed the Envelope Paradox. There are two envelopes, say a green one and a red one, which contain x and 2x dollars; however, you do not know which amount is in which envelope and you do not know x. You are provided at random with one of the envelopes, say you got the green one, you open it and find y. You conclude that the red envelope contains $\frac{2}{1}y$ or 2y, both with a probability of $\frac{1}{2}$. So, on the average, you expect the red envelope to contain $\frac{5}{4}y$ and, being risk neutral, you should agree to exchange the envelopes. The problem is that you could argue the same way after receiving the red envelope: This is called the Envelope Paradox.

Now assume that you have an amount of y dollars which you can change into euros today at an exchange rate of 1 and change it back tomorrow at an exchange rate of a = $\frac{1}{2}$ or 2, both with a probability of $\frac{1}{2}$. Being risk neutral you should do it because you can expect to have $\frac{5}{4}y$ afterwards. The owner of a sum of y euros could argue the same way: That is the essence of the Siegel Paradox (Siegel, 1972). Where is the connection? It is simply that you know that your envelope contains y dollars and you know that the other envelope contains y euros, but you do not know the worth of the euros. Or the other way round.

One might argue that there is no real paradox because, from one point of view, one increases one’s wealth in terms of dollars and from the other point of view in terms of euros and thus there might be no paradox involved in this (Boyer, 1977). But this argument is misleading if there are other sources of exchange rate variability than inflation. Assume that you own a portfolio of y dollars and y euros. You consider as the first option to exchange, today, your dollars into euros and change them back tomorrow. Thus you expect to have tomorrow a new portfolio of ($\frac{5}{4}y$ dollars, y euros). If you choose your second option, namely to exchange your euros today and

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1 Though with a more general distribution of exchange rates.
change them back tomorrow you expect the portfolio (y dollars, \( \frac{5}{4} y \) euros). So, if these are dollars and euros compensated for inflation, both times you increased your expected wealth which is a real paradox.

II. Equivalence of the paradoxes?

The Siegel Paradox is often stated as the impossibility to find a sensible futures market price. In the example above, 1 euro is expected to be worth \( \frac{5}{4} \) dollars tomorrow; thus the dollar price of tomorrow’s euros should be \( \frac{5}{4} \). On the other hand, 1 dollar is expected to be worth \( \frac{5}{4} \) euros tomorrow, so the dollar price of euros should be \( \frac{4}{5} \). What ever exchange rate the futures market chooses you can profit from it. But this is no different a story than we have told without a futures market.

Therefore, is the Siegel Paradox equivalent to the Envelope Paradox? The above description seems to imply such an equivalence, but if one tries to explain (away) the paradox as outlined by Nalebuff (1989) one is faced with a difficulty. Nalebuff (1989) shows that the Envelope Paradox, as stated above, requires infinitely many, equally probable (which is impossible), amounts of money. The paradox falls apart if there is an upper limit\(^2\) of the money in the envelopes and it either vanishes or is connected with an infinite expectation value if one supposes a distribution of an (unlimited) money supply in the envelopes.

Can we conclude that similar assumptions are also hidden in the Siegel Paradox? One way is to ask where the expected wealth increases should come from. There must be other players in the game who - this is the most simple and consistent way to argue - are motivated by the same rationale as I am. They buy or sell tomorrow in order to sell or buy the day after tomorrow, etc. So, it is clear that the most simple consistent extension of the currency game is a world where exchange rates follow a random walk.

III. Speculation when exchange rates or equity prices follow a random walk.

We have seen above that the Siegel Paradox provides us with a simple strategy to exploit exchange rates or equity prices which follow a simple geometric random walk, i.e. rates which are, in every period multiplied by a factor of \( b \) or \( \frac{1}{b} \) with probability \( \frac{1}{2} \). (Below we will generalise the random walk.) Let us now simply combine the two alternative strategies, i.e. starting from a portfolio (y euros, y dollars), in every add period we exchange all the dollars we have to euros and in every even period all the euros we have to dollars. After 2 periods, the expectation value of our new portfolio is \((b_1 \cdot y\text{ euros}, b_1 \cdot y\text{ dollars})\) with

\[
(1) \quad b_1 \cdot \frac{1}{2} a + \frac{1}{2} a > 1 \quad \text{with } a = b^2.
\]

After 2n periods, the expectation value is \((b_n \cdot y\text{ euros}, b_n \cdot y\text{ dollars})\), with

\[
(2) \quad b_n = \sum_{i=0}^{n} a^i \left( \frac{1}{a} \right)^{n-i} \left( \frac{1}{2} \right)^{n-i} \left( \frac{1}{2} \right)^i
\]

\[
= \sum_{i=0}^{n} \left( \frac{a}{2} \right)^i \left( \frac{1}{2} \right)^{n-i} \left( \frac{1}{2} \right)^i
\]

\[
= \sum_{i=0}^{n} x^i \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{n-i} \left( \frac{1}{2} \right)^i
\]

with \( a > 1, x = \frac{a}{2} > 1 - \frac{1}{2a} > 1 \).

As \( x^i > i \) for sufficiently large i,
\( b_n > \sum_{i=0}^{n} \left( 1 - \frac{1}{2a} \right)^i \left( \frac{1}{2a} \right)^{n-i} \left( \frac{n}{i} \right) \)

\[= n \left( 1 - \frac{1}{2a} \right) \]

for sufficiently large \( n \).

Let us now drop the assumption of a bivariate distribution. Let us only assume that the exchange rate \( a \) is drawn from a distribution with \( \text{E}(\log a) = 0 \). (Even a weaker assumption could be used.) It follows

\[ \text{E}(\log \frac{1}{a}) = 0, \]

and from Jensen's inequality

\[ \text{E}(a) > 1, \text{E}(\frac{1}{a}) > 1 \]

\[ \text{E}(a^n) > (\text{E}(a))^n, \text{E}(\left( \frac{1}{a} \right)^n) > \left( \text{E}\left( \frac{1}{a} \right) \right)^n. \]

The value of the assets is \( (\text{E}(a^n), \text{E}(\left( \frac{1}{a} \right)^n)) \) which both increase infinitely for \( n \to \infty \).

Is this really possible?

If you substitute dollars by equity shares and interpret \( a \) as the price of equities, you see that the same strategy is profitable not only in currency markets but in all markets with regular trade - over an infinite number of periods! In option markets or election markets\(^5\) where the asset completely loses its value after a certain date, all evaluation takes place in euros: the Siegel Paradox disappears. So the asset has to have an intrinsic value. In the case of a limited volume of equities the contradiction to owning \( b_n y \) equities in period \( n \) is apparent. So equity prices cannot follow a random walk! And what about currencies?

Where does the increase in speculators' wealth come from? It must come from new market participants, who perhaps enter the market because of the same rationale and with a similar strategy as we once did. Thus, an infinite time horizon is a prerequisite for the Siegel Paradox. If we assume the random walk breaks down at time \( T \), say that from \( T \) on the exchange rate is fixed to \( a^* \), then we could start the same backward induction argument with which Nalebuff (1989) provides us in the case of the Envelope Paradox. There, the profitability to exchange envelopes in both directions vanishes; in the currency game, no transactions are carried out except at a rate of \( a^* \).

Taking the viewpoint of the infinite time horizon or the potentially infinite money supply in the envelopes both problems lead to infinite expectation values. The rejection of such an idea is easier, however, in the case of the one-period envelope problem because, in economics, we are used to getting infinite sums over an infinite time horizon – if we do not discount. A discount rate \( \delta < \frac{1}{x} \) would make the Siegel Paradox disappear because the above strategy is no longer profitable.\(^5\) A discount rate \( \delta > \frac{1}{x} \), however, would leave the paradox untouched but also would provide us with unlimited discounted wealth. The arguments to reject such cases are similar but not identical to those in the envelope case.

First, while in the envelope problem we have no choice of the sums in the envelopes, in the currency game we can choose \( y \) and (if risk neutral) we would prefer to choose it as large as possible. So, from the very beginning all money must be in the game. Second, as mentioned already above, in every period the market has to grow. There have to be entrants who provide the market with the additional volume necessary for successful speculation. But where should this money come from? All this looks like a pyramid game or like a bubble which necessarily has to burst sooner or later.

\(^5\) This discount rate is not the market rate of interest for risk free assets (which we assume to be 0) but has to be regarded in addition to this rate, measuring a personal time preference. The decision to disregard the rate of interest is sensible if we assume that dollars and euros are held as risk free assets between the points of time when the exchange takes place.

\(^4\) See, for example, Brügelambert and Crüger (2002).
Therefore, I conclude that, in a market with rational participants, a random walk of the exchange rates (corrected for inflation) or of equity prices cannot be expected.

Edlin (2002) is the only one to compare the Siegel Paradox with the Envelope Paradox as presented by Nalebuff (1989). Though some of his arguments are similar to those presented here, he concludes (p. 4) that “(n)ot all Siegel profits are illusory Nalebuff profits”. He underpins his statement with a small two-period General Equilibrium model where “Siegel-players” earn profits from “fickle players” who experience a kind of inflation of tastes. This group has to be large enough in order to let Siegel players profit. What does this mean for the currency market? Real profits are possible if exchange rates are determined by random inflation in the two countries involved, if some market participants hold assets the relative value of which is not touched by inflationary processes(?).

Kemp and Sinn (2000) investigate the Siegel Paradox with another kind of equilibrium model. In my eyes, they explain away the paradox: in equilibrium, only risk is traded among the consumers in one country. From the viewpoint of expectation values, the less risk averse agents profit – but that is no paradox.

Contrary to the above models, my arguments are similar to Nalebuff (1989), stressing the contradiction involved in a random walk of exchange rates over an infinite horizon. Though the geometric mean of exchange rates remains 1, the algebraic mean tends to infinity – which corresponds to Nalebuff’s (1989) finding that the mean of the money distribution in the envelopes has to be infinite.

IV. Conclusion

We have seen that the Envelope Paradox is very close to the Siegel Paradox but not completely identical with it. The Envelope Paradox disappears if you exclude the possibility that the amounts in the envelope are randomly drawn from a distribution with an infinite expectation value (see Nalebuff, 1989). The Siegel Paradox disappears if you exclude the possibility that an infinite amount of new money is brought to the market.

Perhaps, the comparison with the Envelope Paradox has shed some light on the nature of the Siegel Paradox. We should not conclude, as some authors do, that the Siegel Paradox is a real phenomenon which opens small but real opportunities of speculation (Sinn, 1989, Kemp and Sinn, 2000, Edlin, 2002, and some of the literature cited there) nor that the usage of logarithmic expectation values is “the solution”. (It is questionable whether such considerable risk aversion exists). One should better conclude that the development of exchange rates and equity prices cannot consistently be described by commonly known random walks.

References


